

Lecture 3 - Linear Model Predictive Control

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LQR - Linear Quadratic regulator

For a discrete time system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

that optimizes the infinite horizon cost

$$V_{LQR} = \min \sum_{k=0}^{\infty} \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k$$

The LQR control law $\mathbf{u} = -K\mathbf{x}$ and the LQR cost $V_{LQR} = \mathbf{x}^T P \mathbf{x}$ is given as the solution to the Discrete Algebraic Riccati Equation (DARE)

$$P = Q + A^T P A - A^T P B H^{-1} B^T P A$$

which gives the LQR gain

$$K = (R + B^T P B)^{-1} B^T P A$$

Constrained Linear Quadratic control

$$\begin{aligned} \min_{\mathbf{w}} \quad & \sum_{k=0}^{\infty} \mathbf{x}_k^{\top} \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^{\top} \mathbf{R} \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A} \mathbf{x}_k + \mathbf{B} \mathbf{u}_k, \quad k = 0, 1, \dots \\ & \mathbf{x}_{lb} \leq \mathbf{x}_k \leq \mathbf{x}_{ub}, \quad k = 0, 1, \dots \\ & \mathbf{u}_{lb} \leq \mathbf{u}_k \leq \mathbf{u}_{ub}, \quad k = 0, 1, \dots \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned}$$

- ▶ No closed-form solution exists!
- ▶ We no longer have a linear control policy $\mathbf{u} = -\mathbf{K}\mathbf{x}$, but a piece-wise affine control policy.
- ▶ Obtained by solving the QP with a finite horizon
- ▶ Problem is strictly convex if $\mathbf{Q} \succeq 0$ and $\mathbf{R} \succ 0$

Constrained Linear Quadratic control

Quadratic cost

$$\sum_{k=0}^{N-1} \left(\frac{1}{2} \mathbf{x}_k^\top Q \mathbf{x}_k + \frac{1}{2} \mathbf{u}_k^\top R \mathbf{u}_k \right) + \frac{1}{2} \mathbf{x}_N^\top P \mathbf{x}_N$$

Linear system dynamics

$$\mathbf{x}_{k+1} = A \mathbf{x}_k + B \mathbf{u}_k$$

Linear constraints

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} x_{ub} \\ -x_{lb} \\ u_{ub} \\ -u_{lb} \end{bmatrix} = D \mathbf{x}_k + E \mathbf{u}_k \leq b$$

Transcription to QP

We need to convert the linear quadratic control problem into a standard QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{x}_N^\top P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = A \mathbf{x}_k + B \mathbf{u}_k, \quad k = 0, 1, \dots, N-1 \\ & D \mathbf{x}_k + E \mathbf{u}_k \leq b, \quad k = 0, 1, \dots, N-1 \\ & D_N \mathbf{x}_N \leq b_N \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top H \mathbf{w} + h^\top \mathbf{w} \\ \text{s.t.} \quad & F^\top \mathbf{w} - f = 0, \\ & G^\top \mathbf{w} - g \leq 0 \end{aligned}$$

Two approaches:

- ▶ Dense condensed (reduced-space)
- ▶ Sparse non-condensed (full-space)

Dense condensed QP

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}}_{:=\mathcal{A}} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{N-2}B & \cdots & AB & B & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}}_{:=\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix}$$

Linear system dynamics

$$\bar{\mathbf{x}} = \mathcal{A}\mathbf{x}(t) + \mathcal{B}\mathbf{w}$$

with $\bar{\mathbf{x}} := [x_0 \ x_1 \ \cdots \ x_{N-1} \ x_N]^\top$ and $\mathbf{w} := [u_0 \ u_1 \ \cdots \ u_{N-1}]^\top$

Dense condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \bar{\mathbf{x}}^\top \bar{\mathbf{Q}} \bar{\mathbf{x}} + \frac{1}{2} \mathbf{w}^\top \bar{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \bar{\mathbf{x}} = \mathcal{A} \mathbf{x}(t) + \mathcal{B} \mathbf{w} \\ & \bar{\mathbf{D}} \bar{\mathbf{x}} + \bar{\mathbf{E}} \mathbf{w} \leq \bar{\mathbf{b}} \end{aligned}$$

with $\bar{\mathbf{Q}} := \begin{bmatrix} I \otimes Q & \\ & P \end{bmatrix}$, $\bar{\mathbf{R}} := I \otimes R$, $\bar{\mathbf{D}} := \begin{bmatrix} I \otimes D & \\ & D_N \end{bmatrix}$, $\bar{\mathbf{E}} := \begin{bmatrix} I \otimes E \\ 0 \end{bmatrix}$,
and $\bar{\mathbf{b}} := \begin{bmatrix} \mathbf{1} \otimes b \\ b_N \end{bmatrix}$.

Substituting $\bar{\mathbf{x}} = \mathcal{A} \mathbf{x}(t) + \mathcal{B} \mathbf{w}$ and rearranging, we get

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w} \\ \text{s.t.} \quad & [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}] \mathbf{w} \leq \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t) \end{aligned}$$

Dense condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w} \\ \text{s.t.} \quad & [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}] \mathbf{w} \leq \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t) \end{aligned}$$

is in the standard QP form

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top \mathbf{H} \mathbf{w} + \mathbf{h}^\top \mathbf{w} \\ \text{s.t.} \quad & \mathbf{G}^\top \mathbf{w} - \mathbf{g} \leq 0 \end{aligned}$$

with

$$\mathbf{H} := [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}]$$

$$\mathbf{h} := [\mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B}]^\top = \mathcal{B}^\top \bar{\mathbf{Q}}^\top \mathcal{A} \mathbf{x}(t)$$

$$\mathbf{G} := [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}]^\top$$

$$\mathbf{g} := \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t)$$

and decision variables $\mathbf{w} := [\mathbf{u}_0 \quad \mathbf{u}_1 \quad \cdots \quad \mathbf{u}_{N-1}]^\top$

Sparse non-condensed QP

Linear system dynamics

$$\begin{aligned}x_0 &= x(t) \\ Ax_0 + Bu_0 - x_1 &= 0 \\ Ax_1 + Bu_1 - x_2 &= 0 \\ &\vdots \\ Ax_{N-1} + Bu_{N-1} - x_N &= 0\end{aligned}$$

Rearranging in matrix form

$$\underbrace{\begin{bmatrix} I & & & & \\ A & B & -I & & \\ & A & B & -I & \\ & & \ddots & \ddots & \ddots \\ & & & A & B & -I \end{bmatrix}}_{:=F^T} \underbrace{\begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}}_{:=w} = \underbrace{\begin{bmatrix} x(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{:=f}$$

Sparse non-condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top H \mathbf{w} \\ \text{s.t.} \quad & F^\top \mathbf{w} - \mathbf{f} = 0, \\ & G^\top \mathbf{w} - \mathbf{g} \leq 0 \end{aligned}$$

$$\text{where } H := \begin{bmatrix} I \otimes \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} & \\ & P \end{bmatrix}, \quad G^\top := \begin{bmatrix} I \otimes [D \quad E] & \\ & D_N \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} \mathbf{1} \otimes \mathbf{b} \\ b_N \end{bmatrix}$$

$$F^\top := \begin{bmatrix} I & & & & & & \\ A & B & -I & & & & \\ & & A & B & -I & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A & B & -I \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{x}(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{w} := [\mathbf{x}_0 \quad \mathbf{u}_0 \quad \mathbf{x}_1 \quad \cdots \quad \mathbf{x}_N]^\top$$

Special case: unconstrained LQ

Transcribe to QP using dense-condensed formulation:

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w}$$

Necessary and sufficient condition for optimality:

$$\begin{aligned} [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathcal{B}^\top \bar{\mathbf{Q}}^\top \mathcal{A} \mathbf{x}(t) &= 0 \\ \Rightarrow \mathbf{w} &= [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}]^{-1} \mathcal{B}^\top \bar{\mathbf{Q}}^\top \mathcal{A} \mathbf{x}(t) \end{aligned}$$

Compare with LQR policy: $\mathbf{u} = -(\mathbf{R} + \mathbf{B}^\top \mathbf{P} \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{A} \mathbf{x}(t)$.

Remarks

- ▶ LQR (recursive approach), dense condensed QP, sparse non-condensed QP - all give same numerical solution for unconstrained problem
- ▶ The QP approaches give an open-loop solution, but the LQR (recursive approach) give a closed loop (feedback) solution
- ▶ Feedback is incorporated in the QP approaches by repeatedly solving it (MPC)
- ▶ Linear constraints can be easily added to the QP

Standard stage cost

General quadratic stage cost

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \frac{1}{2} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \mathbf{d}_x^\top \mathbf{x}_k + \frac{1}{2} \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k + \mathbf{d}_u^\top \mathbf{u}_k$$

Setpoint tracking

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \frac{1}{2} (\mathbf{x}_k - \mathbf{x}_r)^\top \mathbf{Q} (\mathbf{x}_k - \mathbf{x}_r) + \dots = \frac{1}{2} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \underbrace{\mathbf{x}_r^\top \mathbf{Q}}_{:= \mathbf{d}_x^\top} \mathbf{x}_k + \dots$$

Penalize input usage

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \dots + \frac{1}{2} \Delta \mathbf{u}_k^\top \mathbf{S} \Delta \mathbf{u}_k$$

Other stage costs

- ▶ It is also possible to define stage costs with ℓ_1 -norm, though this is not very common.
- ▶ With an ℓ_1 -norm, we do not have a QP, but this can be reformulated as a LP (using auxiliary variables).

$$\min_x \|x\|_1 \quad \Rightarrow \quad \begin{array}{ll} \min_{t,x} & \mathbf{1}^T t \\ \text{s.t.} & -t \leq x \leq t \end{array}$$

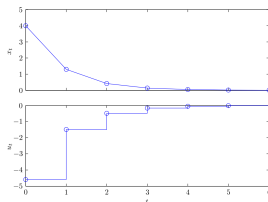
Significance of weights

Weights represent engineering priorities and trade-offs

$$\min \sum_{t=0}^5 q x_{t+1}^2 + r u_t^2$$

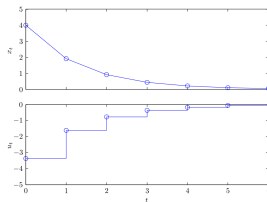
$$\text{s.t.} \quad x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \dots, 5$$

$q = 5, r = 1$



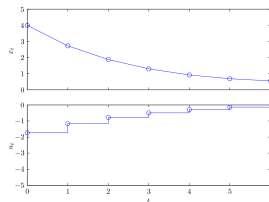
$$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \quad \sum_{t=0}^{N-1} u_t^2 = 23.6$$

$q = 2, r = 1$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \quad \sum_{t=0}^{N-1} u_t^2 = 14.7$$

$q = 1, r = 2$

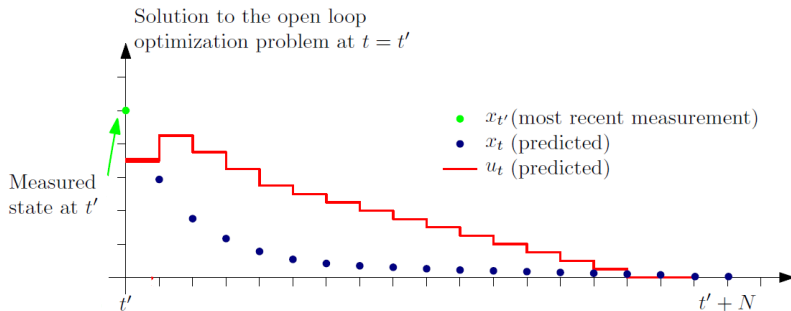


$$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \quad \sum_{t=0}^{N-1} u_t^2 = 5.3$$

Open-loop policy

- Solving QP results in the optimal solution sequence of actions

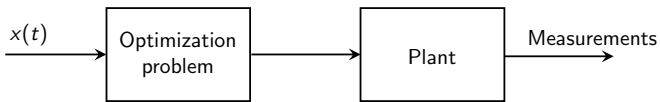
$$[u_0^*, u_1^*, \dots, u_{N-1}^*]^\top$$



- implementing them in open-loop is not robust!

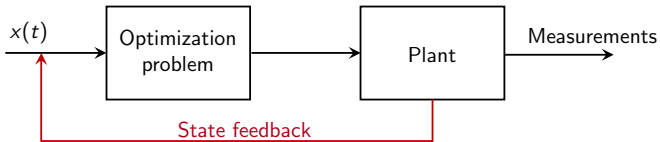
Open-loop policy

$$\pi : \{0, 1, \dots, T - 1\} \rightarrow \mathcal{U}$$



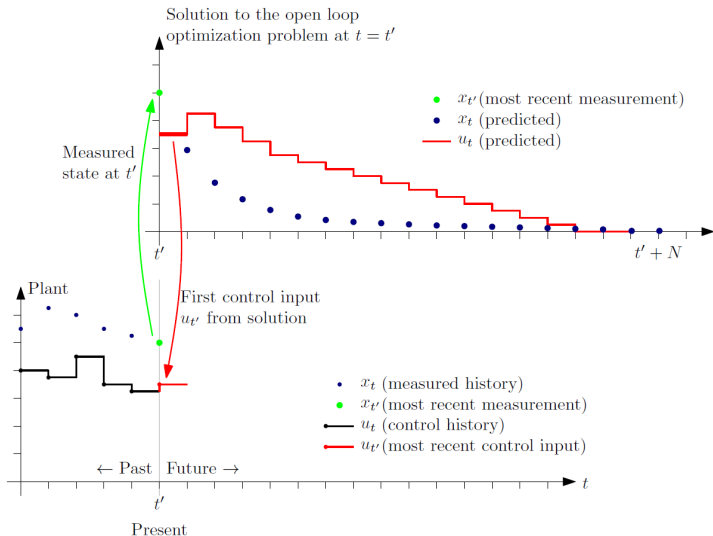
Closed-loop policy

$$\pi : \mathcal{X} \rightarrow \mathcal{U}$$



Model Predictive Control

$$\pi_{\text{mpc}}(x) = u_0^*$$



Model Predictive Control

1. At each time t , solve QP

$$\min_{\mathbf{w}} \mathbf{x}_N^\top P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k$$

$$\text{s.t. } \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad k = 0, 1, \dots$$

$$D\mathbf{x}_k + E\mathbf{u}_k \leq b, \quad k = 0, 1, \dots$$

$$\mathbf{x}_0 = \mathbf{x}(t)$$

to obtain optimal sequence

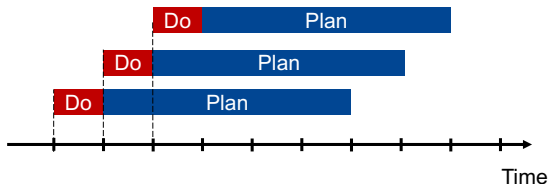
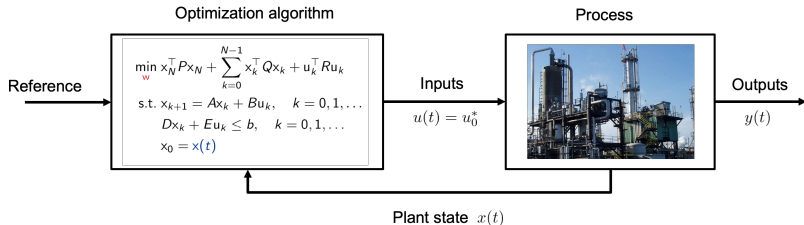
$$[\mathbf{u}_0^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{N-1}^*]^\top$$

2. Implement first action $\mathbf{u}(t) = \pi_{\text{mpc}}(\mathbf{x}(t)) = \mathbf{u}_0^*$
3. Measure next state $\mathbf{x}_{t+1}$, and go to step 1.

This introduces feedback!

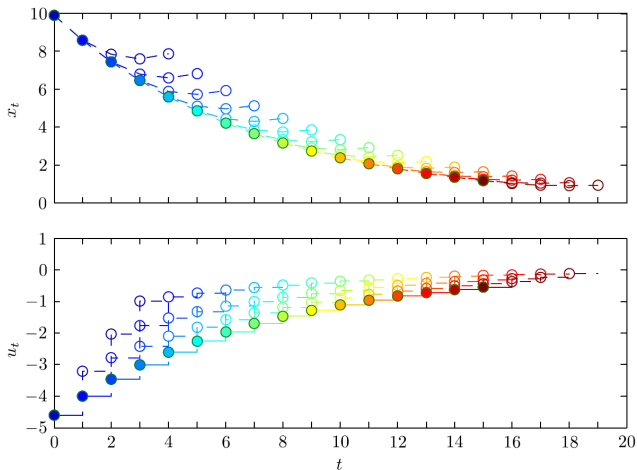


Model Predictive Control

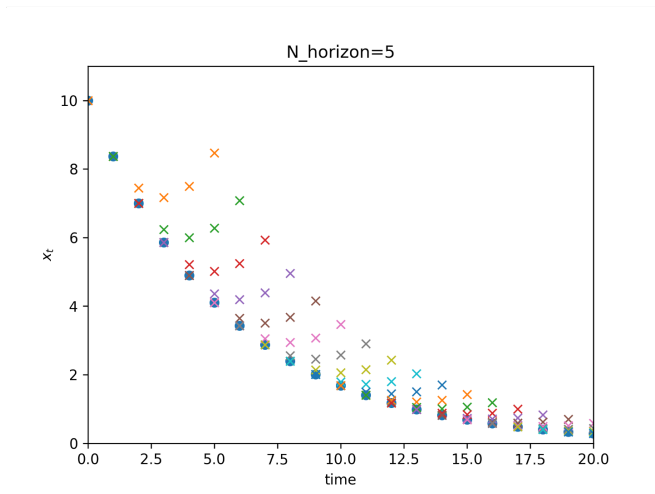


Consequence of finite horizon

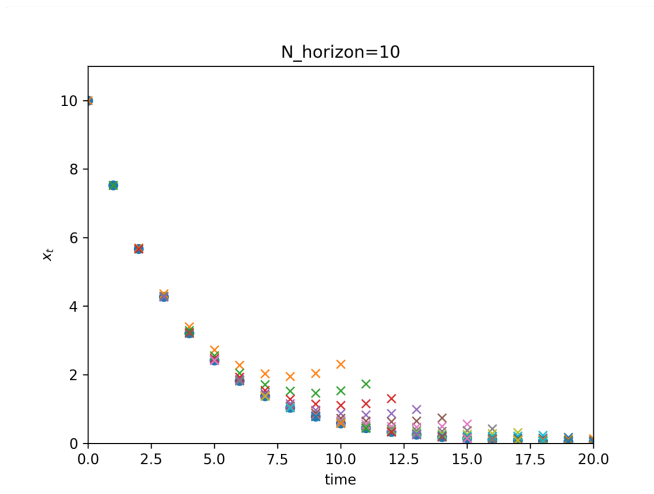
Closed-loop trajectories different from open-loop (optimized) trajectories!



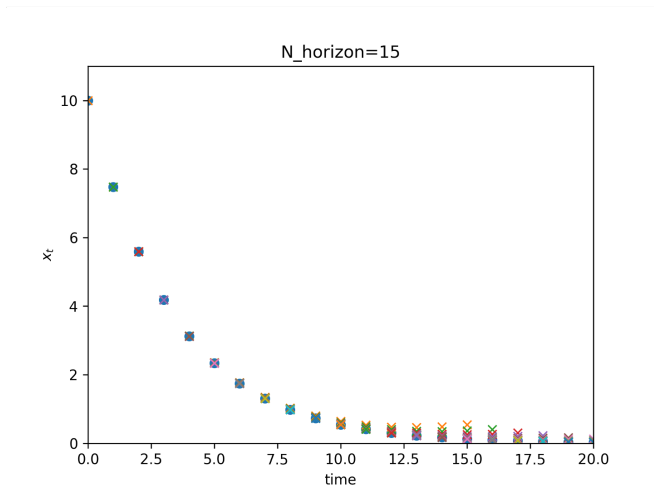
Consequence of finite horizon



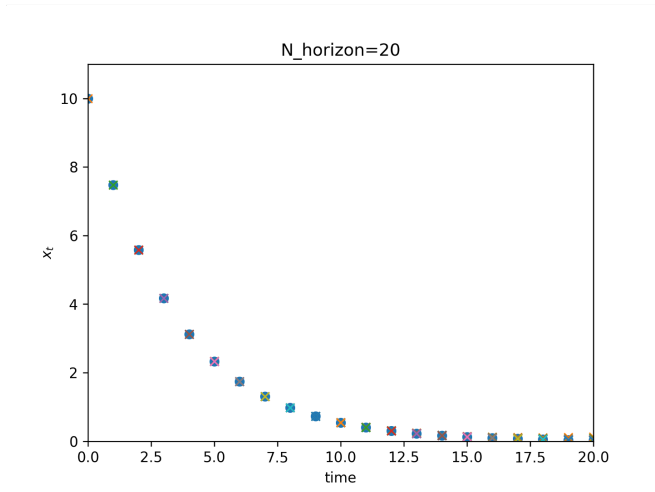
Consequence of finite horizon



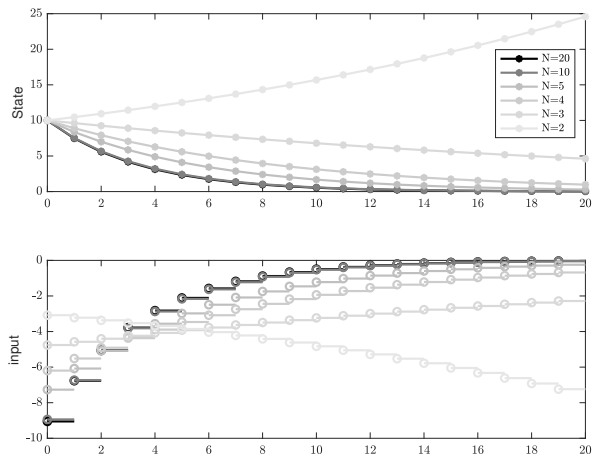
Consequence of finite horizon



Consequence of finite horizon



Closed-loop solution



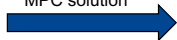
Small horizon can lead to unstable closed-loop performance!!

MPC optimality does not imply stability

$$\min \sum_{t=0}^1 \frac{1}{2} x_{t+1}^2 + \frac{r}{2} u_t^2$$

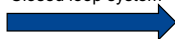
$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1$$

MPC solution

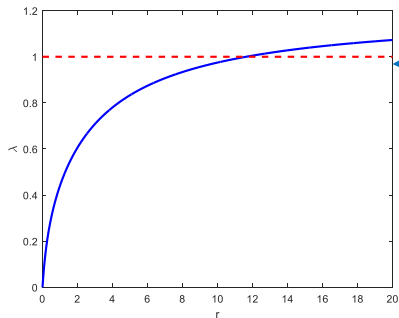


$$u_t = -\frac{1.2 + 2.928r}{1 + 3.44r + r^2} x_t$$

Closed loop system



$$x_{t+1} = \left(1.2 - \frac{1.2 + 2.928r}{1 + 3.44r + r^2} \right) x_t$$



- For given N , design Q and R such that MPC is stable .
- But usually we want to design (tune) Q and R for performance!

Stability of MPC

Infinite horizon: LQR

- ▶ LQR is the solution to the infinite horizon problem

Choose N large enough!

- ▶ Can show theoretically that stability is guaranteed for N large enough, but difficult/conservative to compute this limit
- ▶ Too large N can be computationally expensive (large QP!)

What is *large enough* in practice? (Heuristics)

- ▶ Rule of thumb 1 (simple): Choose N longer than dominating dynamics
- ▶ Rule of thumb 2 (better): Choose N large enough such that open-loop predictions resemble closed-loop (test in simulations!)

Properties of linear MPC

Infinte horizon problem (what we would like to solve)

$$\begin{aligned} \min \quad & \sum_{k=0}^{\infty} \ell(x_k, u_k) \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \quad k = 0, 1, \dots \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \quad k = 0, 1, \dots \\ & x_0 = x(t) \end{aligned}$$

This problem has infinite number of variables - we can't solve it.

Receding horizon problem (what we can solve)

$$\begin{aligned} V_N(x(t)) = \min \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \quad k = 0, 1, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \quad k = 0, 1, \dots, N-1 \\ & x_0 = x(t) \\ & x_N \in \mathcal{X}_f \end{aligned}$$

- ▶ $V_f(x_N)$ approximated the tail of the cost (remaining cost-to-go).
- ▶ $\mathcal{X}_f$ approximates the tail of the constraints.
- ▶ Truncating the prediction horizon can lead to stability and feasibility problems, even without disturbances and plant-model mismatch, due to the short-sightedness.
- ▶ Choice of terminal cost and constraints is important.

Question: How to design $V_f(x_N)$ and $\mathcal{X}_f$ such that we can guarantee desired closed-loop properties?

Stability of linear MPC

Assume

1. Stage cost $\ell(x_k, u_k)$ is positive definite i.e. it is strictly positive and only zero at the origin.
2. Terminal set $\mathcal{X}_f$ is invariant under a local control law $\kappa_f(x)$

$$x_{k+1} = Ax_k + B\kappa_f(x_k) \in \mathcal{X}_f, \quad \forall x_k \in \mathcal{X}_f$$

All state and input constraints are satisfied in $\mathcal{X}_f$

$$\mathcal{X}_f \subseteq \mathcal{X}, \quad \kappa_f(x_k) \in \mathcal{U}, \quad \forall x_k \in \mathcal{X}_f$$

3. Terminal cost is a Lyapunov function in the terminal set $\mathcal{X}_f$, satisfying

$$V_f(x_{k+1}) - V_f(x_k) \leq -\ell(x_k, \kappa_f(x_k)) \quad \forall x_k \in \mathcal{X}_f$$

MPC stability and feasibility

If we choose: $\mathcal{X}_f$ to be an invariant set (Assumption 2) and the terminal cost $V_f(x)$ to be a Lyapunov function with the decrease described in Assumption 3, then

- ▶ The set of feasible initial states $\mathcal{X}_{feas}$ is also the set of initial states which are **recursively feasible** for the closed-loop system under the MPC control law
- ▶ The equilibrium point $(0,0)$ is **asymptotically stable** in the sense of Lyapunov, with $\mathcal{X}_{feas}$ being the region of attraction.
- ▶ $V_N(x)$ is the candidate Lyapunov function for the closed-loop system defined over $\mathcal{X}_{feas}$.

Detailed proof will be given as a part of the nonlinear MPC case.

How to design the terminal ingredients?

Idea: Choose the unconstrained region as our terminal region!

- ▶ Stage cost $\ell(x_k, u_k) = x_k^\top Q x_k + u_k^\top R u_k$, with $Q \succeq 0$, $R \succ 0$
- ▶ Design an unconstrained LQR control law

$$K = -[B^\top P B + R]^{-1} B^\top P A$$

where P is the solution to the Discrete Algebraic Riccati equation (DARE)

$$P = A^\top P A + Q - A^\top P B (B^\top P B + R)^{-1} B^\top P A$$

- ▶ Design the terminal cost as $V_f(x_k) = x_k^\top P x_k$
- ▶ Choose the terminal set $\mathcal{X}_f$ to be the maximal constraint admissible invariant set for the LQR controlled closed-loop.

$$\mathcal{X}_f = \mathcal{O}_\infty := \{x \in \mathcal{X} \mid Kx \in \mathcal{U}\}$$

Verify assumptions hold

1. with $Q \succeq 0$, $R \succ 0$, stage cost $\ell(x_k, u_k)$ is positive definite.
2. By construction, the terminal set is invariant under the LQR control law.
3. With $V_f(x) = x^\top P x$

$$\begin{aligned} & x_{k+1}^\top P x_{k+1} - x_k^\top P x_k \\ & x_k^\top \left(-P + A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A - K^\top R K \right) x_k \\ & - x_k^\top Q x_k - x_k^\top K^\top R K x_k = -\ell(x_k, K x_k) \end{aligned}$$

This is a feasible decrease within the constraint admissible set $\mathcal{X}_f$ (but not outside).

Since all three assumptions are satisfied, the MPC control law is recursively feasible $\forall x_0 \in \mathcal{X}_{feas}$, and the origin is asymptotically stable under the MPC control law, with region of attraction $\mathcal{X}_{feas}$.

Feasible set $\mathcal{X}_{feas}$

- ▶ The size of the feasible set $\mathcal{X}_{feas}$ is determined by the terminal region $\mathcal{X}_f$ and the length of the prediction horizon N .
- ▶ $\mathcal{X}_{feas}$ = set of states that can be steered into $\mathcal{X}_f$ within N steps, i.e., it is the N -step backward reachable set of $\mathcal{X}_f$

$$\mathcal{X}_{feas} = \mathcal{R}_N(\mathcal{X}_f) \subseteq \mathcal{X}$$

Example: an (open-loop) unstable linear system

$$x_{k+1} = \underbrace{\begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix}}_A x_k + \underbrace{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}}_B u_k$$

Constraints

$$|u_k| \leq 1, \quad \begin{bmatrix} -50 \\ -10 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 50 \\ 10 \end{bmatrix}$$

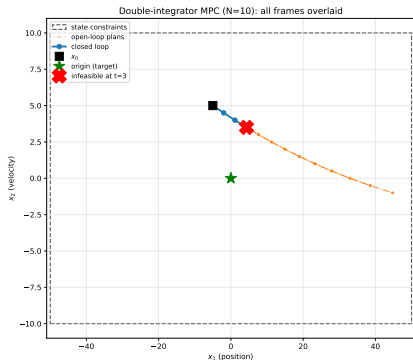
Receding-horizon cost ($N = 10$)

$$\sum_{k=0}^{N-1} (x_k^\top Q x_k + u_k^\top R u_k) + x_N^\top P x_N$$

$$Q = I, \quad R = 1, \quad P = I \text{ (naïve)}$$

With only a quadratic terminal *weight* and **no terminal set**, feasibility is not guaranteed.

The problem: a short horizon loses recursive feasibility



Start at $x_0 = (-5, 5)$

- ▶ Each open-loop plan (orange) is individually feasible and ends inside the box.
- ▶ The controller greedily applies full braking $u = -1$.
- ▶ But $N = 10$ is too short to stop before the wall: at $t = 2$ the OCP is **infeasible**.

Feasible now $\neq$ feasible later. A real controller must always return an input.

The fix: design terminal ingredients

Add a terminal cost $V_f(x) = x^\top P x$ and a terminal set $x_N \in \mathcal{X}_f$ that together make the value function a Lyapunov function and the plan “completable forever”.

Step 1 — LQR (ignore constraints)

Solve the DARE for P ; gain $K = (R + B^\top P B)^{-1} B^\top P A$, closed loop $A_{cl} = A - B K$ (Schur-stable).

$$K = \begin{bmatrix} 0.686 & 0.957 \end{bmatrix}, \quad P = \begin{bmatrix} 2.2747 & 0.3102 \\ 0.3102 & 2.6564 \end{bmatrix}$$

Step 2 — terminal set $\mathcal{X}_f$

Largest set where the LQR law stays admissible *forever*.

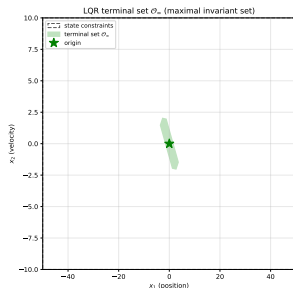
$$\mathcal{X}_f = \{x : C A_{cl}^t x \leq d, \quad t = 0, 1, 2, \dots\}, \quad Cx \leq d : x \in \mathcal{X}, \quad |Kx| \leq u_{\max}.$$

Why $x^\top P x$ is a Lyapunov function on $\mathcal{X}_f$

Along the LQR closed loop, using the DARE and $B^\top P A = (R + B^\top P B)K$:

$$x_{k+1}^\top P x_{k+1} - x_k^\top P x_k = -x_k^\top Q x_k - v_k^\top R v_k < 0, \quad v_k = -K x_k.$$

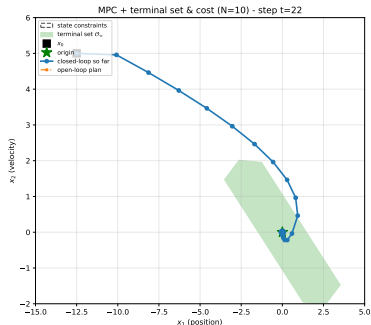
- ▶ $P \succ 0 \Rightarrow V_f$ positive definite.
- ▶ Strict decrease $\Rightarrow$ asymptotic stability.
- ▶ **Valid only on $\mathcal{X}_f$:** there the move $-Kx$ is admissible ($|Kx| \leq u_{\max}$) and $\mathcal{X}_f$ is invariant, so the unconstrained identity is actually realizable.



Result: MPC with terminal set + cost is recursively feasible

OCP now enforces

$$x_N \in \mathcal{X}_f.$$

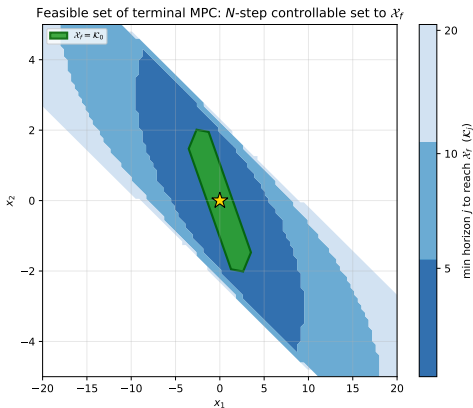


- ▶ Every plan ends in $\mathcal{X}_f$ (green).
- ▶ Shift-the-plan + append LQR move $\Rightarrow$ feasible at $t+1$ whenever feasible at t .
- ▶ Closed loop stays feasible for all time and converges to the origin.

Guarantee bought at a price:
smaller region of attraction
(compare x_0 here vs. Slide 3).

Terminal set shrinks the region of attraction

$$\mathcal{X}_{feas} = \mathcal{R}_N(\mathcal{X}_f) \subseteq \mathcal{X}$$



Takeaways

- ▶ Short-horizon MPC with only a terminal *weight* can be driven from a feasible state into an **infeasible** one.
- ▶ **Terminal ingredients** restore the guarantee: terminal set $\mathcal{X}_f$ (invariance $\Rightarrow$ recursive feasibility) + terminal cost $x^\top P x$ (Lyapunov decrease $\Rightarrow$ stability).
- ▶ Trade-off: terminal sets *shrink the region of attraction* and are expensive to compute in high dimensions.
- ▶ In practice: long horizon + LQR terminal cost, *soft* state constraints (exact penalty $\Rightarrow$ always solvable).
- ▶ $\mathcal{X}_f = 0$ is a simple choice, but region of attraction may become very small.

Additional information from TTK4135

Feasibility

Is there always a feasible solution to the MPC open-loop optimization problem?

- ▶ Not necessarily - state constraints may become infeasible!
- ▶ Soft constraints by adding slack variables

Offset-free MPC - Adding integral action

- ▶ An unmodeled disturbance (plant-model mismatch) will give offset in the controlled variables
- ▶ But, no Integral action in MPC !

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + B_d d_k \\y_k &= Cx_k + C_d d_k \\d_{k+1} &= d_k\end{aligned}$$

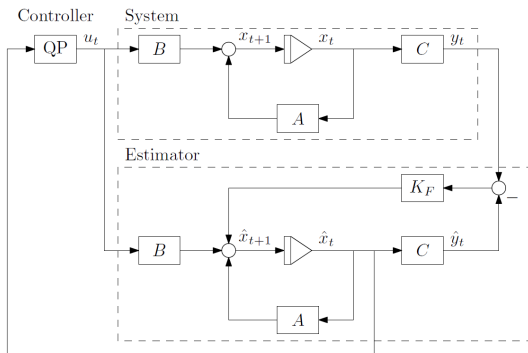
Use the augmented model with a state estimator to estimate $\hat{x}_k$ and $\hat{d}_k$

$$\begin{aligned}\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + w_k \\ y_k &= [C \quad C_d] \begin{bmatrix} x_k \\ d_k \end{bmatrix}\end{aligned}$$

Output feedback MPC

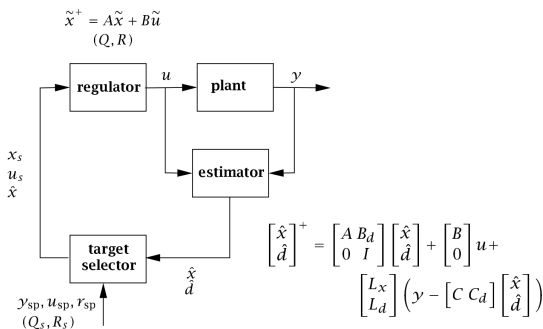
The MPC policy $u(t) = \pi_{\text{mpc}}(x(t))$ is a state-feedback controller!

- Not all states are measured.
- Estimate using Kalman filter / Luenberger observer

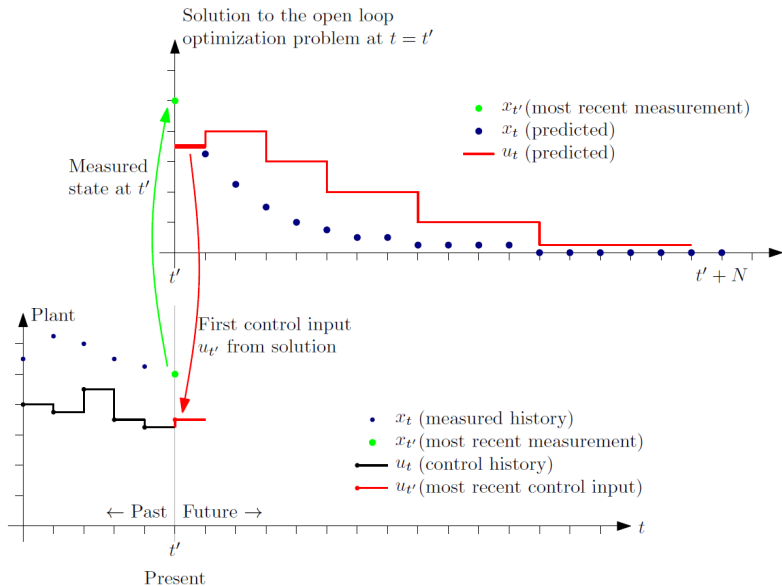


Setpoint tracking

- We want $r_k \rightarrow r_{sp}$, where $r_k = Hy_k$
- NB! controlled variables $r = HCx$ may be different from measured variables $y = Cx$



Complexity reduction: Input blocking



Target Calculator for setpoint tracking

$$\begin{aligned} \min_{\mathbf{x}_s, \mathbf{u}_s} \quad & \frac{1}{2}(\mathbf{C}\mathbf{x}_s - y_{sp})^\top \mathbf{Q}(\mathbf{C}\mathbf{x}_s - y_{sp}) + \frac{1}{2}(\mathbf{u}_s - \mathbf{u}_{iv})^\top \mathbf{R}(\mathbf{u}_s - \mathbf{u}_{iv}) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{H}\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_s \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ r_{sp} \end{bmatrix} \\ & y_{lb} \leq \mathbf{C}\mathbf{x}_s \leq y_{ub} \\ & \mathbf{u}_{lb} \leq \mathbf{u}_s \leq \mathbf{u}_{ub} \end{aligned}$$

Using $(\mathbf{x}_s, \mathbf{u}_s)$, formulate zero-state regulation problem with deviation variables

- ▶ $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_s$
- ▶ $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_s$

When to use MPC instead of PID?

PID is usually sufficient when:

- ▶ Single-input single-output (SISO) systems.
- ▶ Minimal interaction with other control loops.
- ▶ Very low implementation and maintenance complexity required.

MPC becomes preferable when:

- ▶ **Strong coupling between variables:** each input affects multiple outputs.
- ▶ **Multiple objectives must be balanced:** it is not clear how to use the available inputs to meet all goals.
- ▶ **Future events matter:** known reference changes or disturbances can be anticipated.

Summary - Mind map

Open loop policy

$$\{0, 1, \dots\} \rightarrow \mathcal{U}$$

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{\infty} \ell(\mathbf{x}_k, \mathbf{u}_k) \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) \\ & \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k) \leq 0 \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned}$$

(no feedback)



Control Policy $\pi(\mathbf{x})$

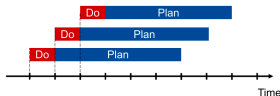
$$\pi : \mathcal{X} \rightarrow \mathcal{U}$$

$$k = N-1, N-2, \dots, 0$$

$$J_k^*(\mathbf{x}_k) = \min_{\mathbf{u}_k \in \mathcal{U}_k(\mathbf{x}_k)} \underbrace{\ell_k(\mathbf{x}_k, \mathbf{u}_k)}_{\text{cost now}} + \underbrace{J_{k+1}^*(f(\mathbf{x}_k, \mathbf{u}_k))}_{\text{cost-to-go}}$$

Compute using
backward DP recursion
(in general intractable, except for LQR)

Model Predictive Control



Summary - Mind map

Choose N to ensure closed-loop system is stable

Choose Q and R to trade-off engineering priorities

$$\min \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + \frac{1}{2} x_N^T P x_N$$

$$\text{s.t. } x_{k+1} = A x_k + B u_k$$

$$x_{lb} \leq x_k \leq x_{ub}$$

$$u_{lb} \leq u_k \leq u_{ub}$$

$$x_0 = x(t)$$

$$x_N \in \mathcal{X}_f$$

LQR cost
from DARE

region where
MPC behaves like LQR

Transcribe to QP

► Dense condensed QP

$$w = [u_0, u_1, \dots, u_{N-1}]^T$$

► Sparse noncondensed QP

$$w = [x_0, u_0, x_1, \dots, u_{N-1}, x_N]^T$$

$$\min_w \frac{1}{2} w^T H w + h^T w$$

$$\text{s.t. } F w = f$$

$$G w \leq g$$

Solve QP
to get u_0^*

• Feasibility

- Soft constraints with slack variables

- $x_{lb} + s \leq x_k \leq x_{ub} + s$ • $s \geq 0$
- Add exact penalty $\rho^T \|s\|_1$

• Setpoint tracking

$$\min_{x_s, u_s} \frac{1}{2} (C x_s - y_{sp})^T Q (C x_s - y_{sp}) + \frac{1}{2} (u_s - u_{iv})^T R (u_s - u_{iv})$$

$$\text{s.t. } \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r_{sp} \end{bmatrix}$$

$$y_{lb} \leq C x_s \leq y_{ub}$$

$$u_{lb} \leq u_s \leq u_{ub}$$

• Disturbance rejection

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix}$$

Formulate zero-state regulation
using deviation variables

State
estimator

measurements
 y

Plant