

Lecture 4 - Nonlinear Model Predictive Control

Dinesh Krishnamoorthy

Department of Engineering Cybernetics, NTNU

Fall 2026

Nonlinear system

- ▶ Consider a discrete time nonlinear time-invariant system described by the nonlinear difference equation

$$x_{k+1} = f(x_k, u_k)$$

Without loss of generality, we can assume $(0, 0)$ is the desired equilibrium with $0 = f(0, 0)$.

- ▶ System is subjected state and input constraints

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}$$

- ▶ Assume full state measured
- ▶ Objective is to drive the system to its origin

Nonlinear MPC

$$\begin{aligned} V_N(x(t)) = \min & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\ \text{s.t. } & x_{k+1} = f(x_k + u_k), & k = 0, 1, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, & k = 0, 1, \dots, N-1 \\ & x_0 = x(t) \\ & x_N \in \mathcal{X}_f \end{aligned}$$

► MPC control law $\pi(x(t)) = u_0^*$

Recap: Linear MPC $\rightarrow$ QP

We converted the linear MPC into a standard QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{x}_N^\top \mathbf{P} \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A} \mathbf{x}_k + \mathbf{B} \mathbf{u}_k, \quad k = 0, 1, \dots, N-1 \\ & \mathbf{D} \mathbf{x}_k + \mathbf{E} \mathbf{u}_k \leq \mathbf{b}, \quad k = 0, 1, \dots, N-1 \\ & \mathbf{D}_N \mathbf{x}_N \leq \mathbf{b}_N \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned} \quad \Rightarrow$$

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top \mathbf{H} \mathbf{w} + \mathbf{h}^\top \mathbf{w} \\ \text{s.t.} \quad & \mathbf{F}^\top \mathbf{w} - \mathbf{f} = 0, \\ & \mathbf{G}^\top \mathbf{w} - \mathbf{g} \leq 0 \end{aligned}$$

Two approaches:

- ▶ Dense condensed (reduced-space)
- ▶ Sparse non-condensed (full-space)

Nonlinear MPC $\rightarrow$ NLP

We need to convert the nonlinear MPC into a standard NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{x}_N^\top P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \quad k = 0, 1, \dots, N-1 \\ & g(\mathbf{x}_k, \mathbf{u}_k) \leq 0, \quad k = 0, 1, \dots, N-1 \\ & \mathbf{x}_N \in \mathcal{X}_f \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{\mathbf{w}} \quad & \phi(\mathbf{w}) \\ \text{s.t.} \quad & g(\mathbf{w}) \leq 0, \\ & h(\mathbf{w}) = 0 \end{aligned}$$

We will look at one approach called, **multiple shooting**, which is analogous to the full-space transcription.

Continuous time model

Modelling system dynamics from first principles results in continuous time Ordinary differential equations (ODE)

- ▶ A continuous time linear system $\dot{x} = A_c x + B_c u$ can be converted to

$$x_{k+1} = A x_k + B u_k$$

using exact discretization (cf. TTK4115 LinSys).

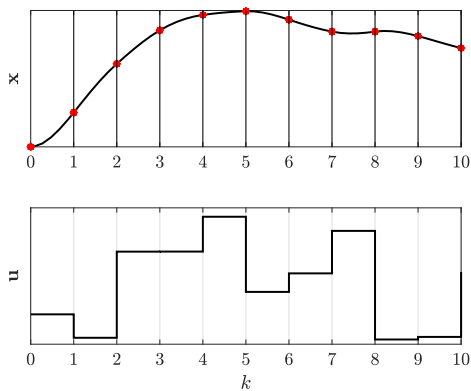
- ▶ A nonlinear ODE of the form $\dot{x} = f_c(x, u)$ must be converted to a nonlinear difference equation

$$x_{k+1} = f(x_k, u_k)$$

using integrators.

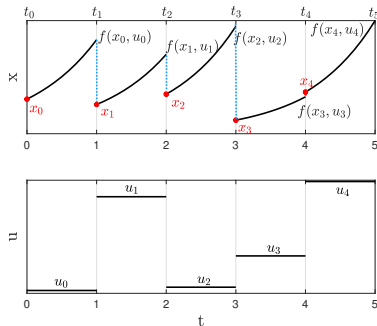
NLP transcription

Decision variables: $\mathbf{w} = [x_0, u_0, x_1, u_1, \dots, x_{N-1}, u_{N-1}, x_N]^\top$



NLP transcription

Decision variables: $\mathbf{w} = [x_0, u_0, x_1, u_1, \dots, x_{N-1}, u_{N-1}, x_N]^\top$



- ▶ x_k is a decision variable that the optimizer can choose
- ▶ But x_{k+1} must be given by the integrator

$$x_{k+1} - f(x_k, u_k) = 0, \\ k = 0, \dots, N-1$$

Enforce this as equality constraints in the NLP

Integrators

Consider the ODE with defined initial conditions $x(0) = x_0$

$$\dot{x}(t) = f_c(x(t), t), \quad t \in [0, T]$$

► Forward Euler

$$x_{k+1} = x_k + \underbrace{\Delta t f_c(x_k, t_k)}_{=f(x_k)}$$

► 4th order Runge-Kutta

$$x_{k+1} = x_k + \underbrace{\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}_{=f(x_k)}$$

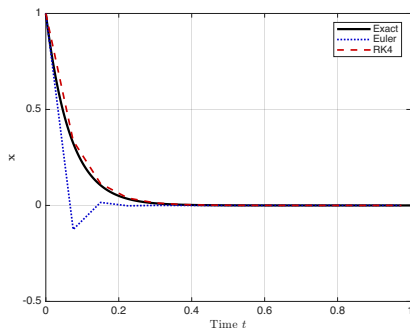
$$k_1 = \Delta t f_c(x_k, t_k)$$

$$k_2 = \Delta t f_c\left(x_k + \frac{k_1}{2}, t_k + \frac{\Delta t}{2}\right)$$

$$k_3 = \Delta t f_c\left(x_k + \frac{k_2}{2}, t_k + \frac{\Delta t}{2}\right)$$

$$k_4 = \Delta t f_c(x_k + k_3, t_k + \Delta t)$$

$$\dot{x} = -Kx$$



Nonlinear MPC

Decision variables: $\mathbf{w} = [x_0, u_0, x_1, u_1, \dots, x_{N-1}, u_{N-1}, x_N]^\top$

$$\begin{aligned} V_N(x(t)) = \min_{\mathbf{w}} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\ \text{s.t.} \quad & x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, 1, \dots, N-1 \\ & x_0 = x(t) \\ & x_N \in \mathcal{X}_f \end{aligned}$$

► MPC control law $\pi(x(t)) = u_0^*$

How to design a nonlinear MPC such that the equilibrium point is asymptotically stable under the MPC control law?

Nonlinear MPC design

A1. Assume continuity of system and cost

The functions $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n$, $\ell : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$,
 $V_f : \mathcal{X}_f \rightarrow \mathbb{R}_{\geq 0}$ are continuous, and

$$f(0,0) = 0, \quad \ell(0,0) = 0, \quad V_f(0) = 0$$

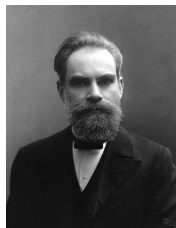
A2. Assume properties of the constraint sets

The set $\mathcal{Z} := \mathcal{X} \times \mathcal{U}$ is closed. $\mathcal{U}$ is compact and is uniformly bounded in $\mathcal{X}$. The set $\mathcal{X}_f \subseteq \mathcal{X}$ is compact, and each set contains the origin.

Under assumptions A1 and A2, we can guarantee that a solution to the optimal control problem exists.

Intermezzo: Stability theory

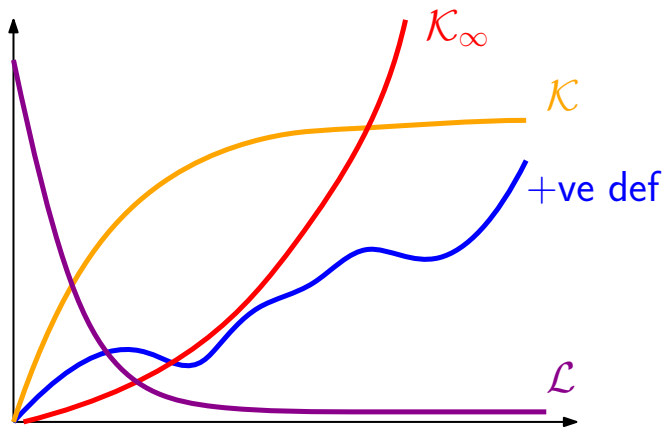
- ▶ For physical systems that does not generate energy internally and with no external input, one would expect the system to naturally reach some kind of rest configuration asymptotically, simply because its energy will be dissipated.
- ▶ The idea of Lyapunov was to adopt such energy-like functions also to test stability



Comparison functions

- ▶ A function $\alpha : \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+}$ is said to be **positive definite** if its continuous $\alpha(0) = 0$ and $\alpha(x) > 0, \forall x > 0$
- ▶ A function $\alpha : \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+}$ is said to be class $\mathcal{K}$ if positive definite and increasing.
- ▶ A function $\alpha : \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+}$ is said to be class $\mathcal{K}_{\infty}$ if class $\mathcal{K}$ and unbounded.
- ▶ A function $\lambda : \mathbb{R}_{0+} \rightarrow \mathbb{R}_{0+}$ is said to be class $\mathcal{L}$ if continuous, $\lambda(0) > 0$, decreasing, $\lim_{t \rightarrow \infty} \lambda(t) = 0$.
- ▶ A function $\beta : \mathbb{R}_{0+}^2 \rightarrow \mathbb{R}_{0+}$ is said to be class $\mathcal{KL}$ if continuous, and $\beta(\cdot, t)$ is class $\mathcal{K} \forall t > 0$, and $\beta(x, \cdot)$ is class $\mathcal{L} \forall x > 0$.

Comparison functions



Lyapunov stability framework

Suppose $\mathcal{X}$ is a positive invariant set for $x^+ = f(x)$, a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function in $\mathcal{X}$ for $x^+ = f(x)$, if

- There exists functions $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that

$$V(x) \geq \alpha_1(|x|)$$

$$V(x) \leq \alpha_2(|x|)$$

i.e., V is upper and lower bounded.

- There exists a positive definite function α_3 such that for any $x \in \mathcal{X}$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|)$$

i.e., V is decreasing if it is not at the origin.

Lyapunov Stability theorem

Suppose there exists a Lyapunov function in $\mathcal{X}$ for the system $x^+ = f(x)$, then the origin is asymptotically stable in $\mathcal{X}$ for $x^+ = f(x)$.

- ▶ if $\mathcal{X} \subset \mathbb{R}^n$ - Locally asymptotically stable
- ▶ if $\mathcal{X} = \mathbb{R}^n$ - Globally asymptotically stable

If in addition, there exists some positive constants c_i and a for $i = 1, 2, 3$ such that $\alpha_i(x) = c_i|x|^a$, the the origin is exponentially stable.

Back to MPC design: show the existence of a Lyapunov function for the closed loop system under the MPC control law.

Nonlinear MPC design

$$\begin{aligned} V_N(x(t)) = \min \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\ \text{s.t.} \quad & x_{k+1} = f(x_k + u_k), \quad k = 0, 1, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, 1, \dots, N-1 \\ & x_0 = x(t) \\ & x_N \in \mathcal{X}_f \end{aligned}$$

- ▶ MPC control law $\pi(x(t)) = u_0^*$
- ▶ The idea is to show that the value function $V_N(\cdot)$ is a valid Lyapunov function for the closed loop system

$$x^+ = f(x, \pi(x)), \quad \forall x \in \mathcal{X}_{feas}$$

- ▶ First, we must show that $\mathcal{X}_{feas}$ is positively invariant $\Rightarrow$ recursive feasibility

Nonlinear MPC design

For stability and recursive feasibility, we need to design the terminal cost and terminal set carefully.

Design terminal ingredients that satisfies the following assumption

A3. Basic stability assumption

$V_f(\cdot)$, $\mathcal{X}_f$, and $\ell(\cdot)$ are designed to have the following properties

- For all $x \in \mathcal{X}_f$, there exists a feasible u satisfying

$$f(x, u) \in \mathcal{X}_f$$

$$V_f(f(x, u)) - V_f(x) \leq -\ell(x, u)$$

- there exists $\alpha_1(\cdot)$, $\alpha_f(\cdot) \in \mathcal{K}_\infty$ satisfying

$$\ell(x, u) \geq \alpha_1(|x|) \quad \forall x \in \mathcal{X}_{feas}$$

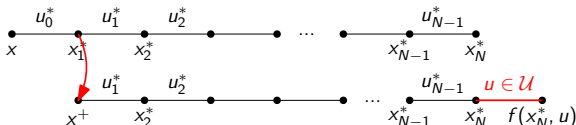
$$V_f(x) \leq \alpha_f(|x|) \quad \forall x \in \mathcal{X}_f$$

Recursive feasibility

- Take any $x \in \mathcal{X}_{feas}$, solving the MPC problem returns the optimal input and state sequence

$$\{u_0^*, u_1^*, \dots, u_{N-1}^*\} \text{ and } \{x, x_1^*, \dots, x_{N-1}^*, x_N^*\}$$

with $x_N^* \in \mathcal{X}_f$



- The successor state is given by $x^+ = f(x, u_0^*) = f(x, \pi(x))$
- The new optimal input and state sequence given by

$$\{u_1^*, \dots, u_{N-1}^*, u\} \text{ and } \{x^+, x_2^*, \dots, x_{N-1}^*, x_N^*, f(x_N^*, u)\}$$

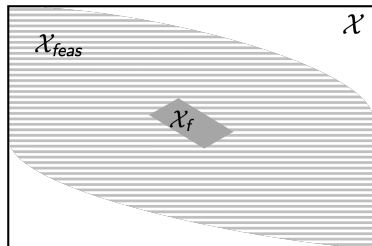
is hence feasible.

- Hence $x^+ \in \mathcal{X}_{feas}$. Since $x \in \mathcal{X}_{feas}$ was chosen arbitrarily, $\mathcal{X}_{feas}$ is positively invariant for $x^+ = f(x, \pi(x))$

NMPC stability

Basic stability assumption (A3) says that the terminal cost $V_f(x)$ is a Lyapunov function within the terminal set $\mathcal{X}_f$, i.e., for $x \in \mathcal{X}_f$, it satisfies

$$\begin{aligned}V_f(f(x, u)) - V_f(x) &\leq -\ell(x, u) \\ \ell(x, u) &\geq \alpha_1(|x|) \\ V_f(x) &\leq \alpha_f(|x|)\end{aligned}$$



To show closed-loop system is stable with region of attraction $\mathcal{X}_{feas}$, we need to show that the value function satisfies:

1. $V_N(x) \geq \alpha_1(|x|)$
2. $V_N(x) \leq \alpha_2(|x|)$
3. $V_N(x^+) - V_N(x) \geq -\alpha_3(|x|)$

1. Lower bound

- ▶ The value function $V_N(x) \geq \ell(x, \pi(x))$ for all $x \in \mathcal{X}_{feas}$
- ▶ Since $\ell(x, u) \geq \alpha_1(|x|)$ for all $x \in \mathcal{X}_{feas}$ and feasible input (A3), we have

$$V_N(x) \geq \alpha_1(|x|)$$

2. Upper bound

If $\mathcal{X}_f$ contains an interior, we can establish the upper bound

- From Basic stability assumption (A3), we have

$$\ell(x, u) \leq V_f(x) - V_f(f(x, u))$$

- Taking a telescoping sum, we get

$$\begin{aligned} \sum_{k=0}^{N-1} \ell(x_k, u_k) &\leq V_f(x_0) - V_f(x_N) \\ \underbrace{\sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N)}_{=V_N(x, \tilde{\mathbf{u}})} &\leq V_f(x_0) \leq \alpha_f(|x|) \end{aligned}$$

- Since the MPC policy can only do better than the local control law

$$V_N(x) \leq V_N(x, \tilde{\mathbf{u}}) \leq V_f(x_0) \leq \alpha_f(|x|)$$

2. Upper bound (via weak controllability)

If $\mathcal{X}_f$ does not have an interior, we can assume the upper bound

- ▶ Weak controllability assumption: For all $x \in \mathcal{X}_{feas}$, there exists $\alpha_2 \in \mathcal{K}_\infty$ such that

$$V_N(x) \leq \alpha_2(|x|)$$

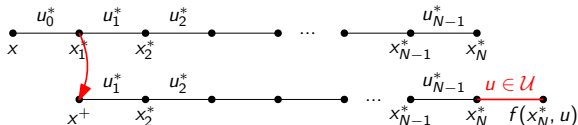
- ▶ This assumption states that states x can be steered to $\mathcal{X}_f$ and the cost of doing so is not excessive.

3. Descent condition

- For any $x \in \mathcal{X}_{feas}$, solving the MPC problem returns the minimizing sequence

$$\mathbf{u}_t := \{u_0^*, u_1^*, \dots, u_{N-1}^*\} \text{ with cost } V_N(x)$$

with $x_N^* \in \mathcal{X}_f$



- The successor state is given by $x^+ = f(x, u_0^*) = f(x, \pi(x))$
- At the next time step, MPC yields the minimizing sequence

$$\mathbf{u}_{t+1} := \{u_0^*, u_1^*, \dots, u_{N-1}^*\} \text{ with cost } V_N(x)$$

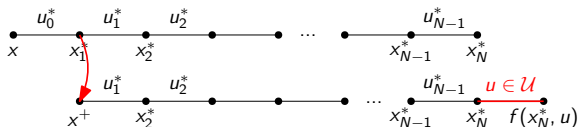
- But let's consider any feasible sequence, with the last term replaced with a feasible input

$$\tilde{\mathbf{u}} := \{u_0^*, u_1^*, \dots, u\} \text{ with cost } V_N(x^+, \tilde{\mathbf{u}})$$

- Since $\tilde{\mathbf{u}}$ is suboptimal,

$$V_N(x^+) \leq V_N(x^+, \tilde{\mathbf{u}})$$

3. Descent condition



$$\begin{aligned}
 V_N(x) &= \ell(x, \pi(x)) + \ell(x_1^*, u_1^*) + \ell(x_2^*, u_2^*) + \cdots + \ell(x_{N-1}^*, u_{N-1}^*) + V_f(x_N^*) \\
 V_N(x^+, \tilde{u}) &= \ell(x_1^*, u_1^*) + \ell(x_2^*, u_2^*) + \cdots + \ell(x_{N-1}^*, u_{N-1}^*) \\
 &\quad + \ell(x_N^*, u) + V_f(f(x_N^*, u))
 \end{aligned}$$

► Cost decrease

$$V_N(x^+, \tilde{u}) - V_N(x) = \underbrace{\ell(x_N^*, u) + V_f(f(x_N^*, u)) - V_f(x_N^*)}_{\leq 0} - \ell(x, \pi(x))$$

$$V_N(x^+) \leq V_N(x^+, \tilde{u}) = V_N(x) - \ell(x, \pi(x))$$

$$V_N(x^+) - V_N(x) \leq -\ell(x, \pi(x))$$

► Since $\ell(x, u)$ is positive definite,

$$V_N(x^+) - V_N(x) \leq -\alpha_1(|x|)$$

Do we really need a terminal constraint set?

Recall the two terminal ingredients: the terminal **cost** $V_f(x_N)$ and the terminal **constraint** $x_N \in \mathcal{X}_f$.

- ▶ Adding the terminal cost V_f is **cheap**: just an extra term in the objective, it does not change the structure of the OCP.
- ▶ Adding the terminal constraint $x_N \in \mathcal{X}_f$ is a **state constraint**, which
 - ▶ makes the optimization harder (extra, often nonlinear, state constraint),
 - ▶ shrinks the feasible set $\mathcal{X}_{feas}$,
 - ▶ requires computing $\mathcal{X}_f$ explicitly – difficult for nonlinear systems.
- ▶ Without state constraints one can establish **continuity of $V_N(\cdot)$** $\Rightarrow$ inherent robustness.

Can we drop the terminal constraint $x_N \in \mathcal{X}_f$ and still guarantee asymptotic stability?

Do we really need a terminal constraint set?

Recall the two terminal ingredients: the terminal **cost** $V_f(x_N)$ and the terminal **constraint** $x_N \in \mathcal{X}_f$.

- ▶ Adding the terminal cost V_f is **cheap**: just an extra term in the objective, it does not change the structure of the OCP.
- ▶ Adding the terminal constraint $x_N \in \mathcal{X}_f$ is a **state constraint**, which
 - ▶ makes the optimization harder (extra, often nonlinear, state constraint),
 - ▶ shrinks the feasible set $\mathcal{X}_{feas}$,
 - ▶ requires computing $\mathcal{X}_f$ explicitly – difficult for nonlinear systems.
- ▶ Without state constraints one can establish **continuity of $V_N(\cdot)$** $\Rightarrow$ inherent robustness.

Can we drop the terminal constraint $x_N \in \mathcal{X}_f$ and still guarantee asymptotic stability?

Eliminating the terminal constraint: the idea

Idea: keep the terminal cost, drop the terminal constraint, and instead restrict the initial state to a sublevel set of the value function. The terminal constraint is then satisfied **implicitly**.

- ▶ Let the terminal set be a sublevel set of V_f : $\mathcal{X}_f = \{x : V_f(x) \leq a\}$, $a > 0$.
- ▶ Terminally **unconstrained** MPC with terminal weight $\beta \geq 1$:

$$\begin{aligned}\hat{V}_N(x(t)) = \min \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + \beta V_f(x_N) \\ \text{s.t.} \quad & x_{k+1} = f(x_k, u_k), & k = 0, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, & k = 0, \dots, N-1 \\ & x_0 = x(t) & \text{(no } x_N \in \mathcal{X}_f\text{)}\end{aligned}$$

- ▶ A3 still holds (V_f is a local CLF on $\mathcal{X}_f$). $\beta = 1$ recovers the plain unconstrained problem.
- ▶ MPC control law $\hat{\pi}(x(t)) = u_0^*$.

Implicit terminal constraint: entering the terminal region

Extra assumption (A4): the stage cost is bounded below outside $\mathcal{X}_f$,

$$\ell(x, u) \geq d > 0 \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_f, u \in \mathcal{U}.$$

Lemma (entering the terminal region). Let $\{x_0^*, \dots, x_N^*\}$ be the optimal state trajectory of the terminally unconstrained problem. If $x_N^* \notin \mathcal{X}_f$, then $x_k^* \notin \mathcal{X}_f$ for all $k = 0, \dots, N - 1$.

Budget argument. If $x_N^* \notin \mathcal{X}_f$, then every stage is spent outside $\mathcal{X}_f$ ($\ell \geq d$) and $V_f(x_N^*) > a$, hence

$$\hat{V}_N(x) = \sum_{k=0}^{N-1} \ell(x_k^*, u_k^*) + \beta V_f(x_N^*) > Nd + \beta a.$$

Contrapositive: $\hat{V}_N(x) \leq Nd + \beta a \Rightarrow x_N^* \in \mathcal{X}_f$ (terminal constraint holds implicitly).

Stability without a terminal constraint

Define the sublevel set of the value function

$$\Gamma_N := \{x \in \mathcal{X} : \hat{V}_N(x) \leq Nd + \beta a\}.$$

Theorem. Under A1–A4 with $\mathcal{X}_f = \{x : V_f(x) \leq a\}$, the origin is asymptotically stable for the closed-loop system $x^+ = f(x, \hat{\pi}(x))$ with region of attraction Γ_N , and Γ_N is positively invariant.

- ▶ For every $x \in \Gamma_N$, the optimal terminal state lies in $\mathcal{X}_f$ (implicit terminal constraint).
- ▶ Hence on Γ_N the problem behaves exactly like the terminally *constrained* one $\Rightarrow$ the earlier Lyapunov analysis (lower bound, upper bound, descent) carries over to $\hat{V}_N(\cdot)$.

Role of the terminal weight β on the region of attraction

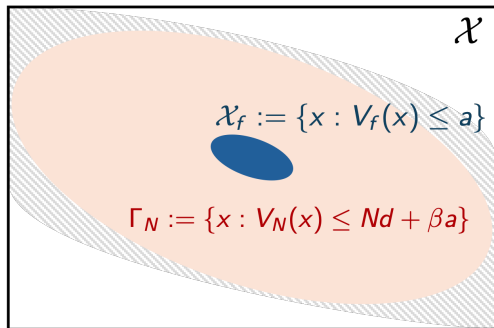
The region of attraction $\Gamma_N = \{x : \hat{V}_N(x) \leq Nd + \beta a\}$ depends on the terminal weight β .

- ▶ **Monotonicity:** Γ_N grows with β , $\beta_1 \leq \beta_2 \Rightarrow \Gamma_N^{\beta_1} \subseteq \Gamma_N^{\beta_2}$.
- ▶ **Reachability:** for any x that can be feasibly steered into the interior of $\mathcal{X}_f$, there exists a β large enough such that $x \in \Gamma_N$.

Intuition: a larger β penalizes the terminal state more heavily $\Rightarrow$ the optimizer works harder to steer x_N into $\mathcal{X}_f \Rightarrow$ the implicit-terminal-constraint region Γ_N enlarges.

Role of the terminal weight β on the region of attraction

- β is a **tuning knob**: increasing β enlarges the region of attraction (towards the maximal stabilizable set), at the cost of a heavier terminal penalty.



Is a terminal constraint necessary?

- ▶ **For stability:** *not* necessary – keep V_f (scaled by β) and restrict $x_0 \in \Gamma_N$; the terminal constraint holds implicitly.
- ▶ **Benefits:** easier optimization, larger and simpler feasible region, continuity of $V_N(\cdot) \Rightarrow$ inherent robustness.
- ▶ **Caveat:** Γ_N is characterized only *implicitly*; if nested feasible sets / standard recursive feasibility are required, a *control-invariant* terminal constraint is necessary (Mayne, 2013).

Summary: the terminal *cost* drives stability; the terminal *constraint* can be replaced by the implicit constraint $x_0 \in \Gamma_N$, with β controlling the size of the region of attraction.