

Lecture 5 - Suboptimal Model Predictive Control

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From optimal to suboptimal MPC

- ▶ In Lecture 4, asymptotic stability relied on solving the MPC problem to **global optimality** at every step:

$$V_N(x) = \min_u V_N(x, u).$$

- ▶ For **nonlinear** systems the OCP is **nonconvex** $\Rightarrow$ the global minimum generally **cannot** be computed within the sampling time.
- ▶ Or solver iterations not completed within the sampling time
- ▶ Yet the stability proof *assumed* the global optimum is attained.

Key observation: the warm start already gives the decrease

Recall the descent step from Lecture 4. The shifted candidate sequence

$$\tilde{\mathbf{u}} = \{u_1^*, \dots, u_{N-1}^*, \kappa_f(x_N^*)\}$$

is feasible for x^+ and, using A3, satisfies

$$V_N(x^+, \tilde{\mathbf{u}}) \leq V_N(x) - \ell(x, \pi(x)).$$

- ▶ No optimization is required to obtain this cost decrease – the shift plus the terminal control law κ_f already deliver it.
- ▶ In practice: run a few iterations of the optimizer to *improve* on $\tilde{\mathbf{u}}$; if even that fails, simply return $\tilde{\mathbf{u}}$.
- ▶ A cheap, stabilizing controller that places very low demands on the optimizer.

The obstacle: V_N is not a function of x alone

- ▶ For a given state x there are **many** feasible sequences $\tilde{u}$, each with a different cost $V_N(x, \tilde{u})$.
- ▶ The suboptimal controller can generate **many** different trajectories from the same x_0 .
- ▶ Hence the cost along the algorithm, $x \mapsto V_N(x, u)$, is **not** a function of x alone and cannot play the role of the Lyapunov function $V(x)$ as before.

Resolution: enlarge the notion of state to include the warm start.

Extended state

- Treat the pair

$$z := (x, \tilde{u}) \quad (\text{state} + \text{warm start})$$

as the state of the closed loop.

- Suboptimal MPC is then the evolution of the **extended state**, a set-valued (**difference inclusion**) system

$$z^+ \in H(z),$$

set-valued because the suboptimal control law is not unique.

Extended state

The extended state evolves according to the following difference equation

$$z^+ \in H(z) := \left\{ (x^+, \tilde{u}^+) \left| \begin{array}{l} x^+ = f(x, u) \\ \tilde{u}^+ = \xi(x, u) \end{array} \right. \right\}$$

where $\xi(x, u) = \{u_1, \dots, u_{N-1}, \kappa_f(x_N)\}$ is the shifted warm start.

Suboptimal control law and successor warm start

Given the pair $(x, \tilde{\mathbf{u}})$:

- ▶ **Admissible control sequences** are the feasible ones that do not increase the cost:

$$\tilde{\mathcal{U}}_N(x, \tilde{\mathbf{u}}) := \{u \in \mathcal{U}_N(x) : V_N(x, u) \leq V_N(x, \tilde{\mathbf{u}})\}.$$

- ▶ **Suboptimal control law**: $\pi(x, \tilde{\mathbf{u}}) = u_0$, the first element of any $u \in \tilde{\mathcal{U}}_N(x, \tilde{\mathbf{u}})$.
- ▶ **Successor warm start**: shift and append the terminal control law,

$$\tilde{\mathbf{u}}^+ = \{u_1, \dots, u_{N-1}, \kappa_f(x_N)\}.$$

The control law depends on **both** x and the warm start $\tilde{\mathbf{u}}$.

Suboptimal MPC algorithm

Offline: choose $\mathcal{X}_f, V_f$ satisfying A3. Get initial $x \in \mathcal{X}_{feas}$ and any admissible warm start $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$.

Online, repeat:

1. Measure the current state x .
2. Compute *any* u with $V_N(x, u) \leq V_N(x, \tilde{\mathbf{u}})$ (zero optimizer iterations is allowed).
3. Inject the first element $u_0 = \pi(x, \tilde{\mathbf{u}})$.
4. Form the successor warm start $\tilde{\mathbf{u}}^+ = \{u_1, \dots, u_{N-1}, \kappa_f(x_N)\}$.

Step 2 places very low demands on the optimizer – it only needs to not increase the cost relative to the warm start.

Extended-state dynamics

- ▶ The extended state $z = (x, \tilde{\mathbf{u}})$ evolves as the **difference inclusion**

$$z^+ \in H(z), \quad x^+ = f(x, \pi(x, \tilde{\mathbf{u}})).$$

- ▶ Analysis takes place on the **extended state space**

$$\mathcal{Z}_N := \{(x, \tilde{\mathbf{u}}) : x \in \mathcal{X}_{feas}, \tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)\}.$$

Linking Lemma. There exists $r \in \mathcal{K}_\infty$ such that $|\tilde{\mathbf{u}}| \leq r(|x|)$ for all $(x, \tilde{\mathbf{u}}) \in \mathcal{Z}_N$.

This lets a bound on the extended state z be converted into a bound on x .

Lyapunov function for difference inclusions

For $z^+ \in H(z)$ with $H(0) = \{0\}$, V is a **Lyapunov function** on a positively invariant set $\mathcal{Z}$ if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for all $z \in \mathcal{Z}$

$$\begin{aligned}\alpha_1(|z|) &\leq V(z) \leq \alpha_2(|z|) \\ \sup_{z^+ \in H(z)} V(z^+) - V(z) &\leq -\alpha_3(|z|)\end{aligned}$$

If such a V exists on a positively invariant $\mathcal{Z}$ containing the origin, the origin is asymptotically stable in $\mathcal{Z}$ for $z^+ \in H(z)$.

The descent must hold for **all** successors $z^+ \in H(z)$ (set-valued map).

Stability of suboptimal MPC

Theorem. Under A1–A3, with $\ell(x, u) \geq \alpha_1(|(x, u)|)$ and $\mathcal{X}_f = \{x : V_f(x) \leq b\}$ a sublevel set of V_f , the function

$$V_N(z) = V_N(x, \tilde{u})$$

is a Lyapunov function for the extended system $z^+ \in H(z)$ on $\mathcal{Z}_N$. Hence the origin is asymptotically stable in $\mathcal{Z}_N$.

Proof ingredients:

- ▶ **Lower bound:** $V_N(z) \geq \alpha_1(|z|)$ from the stage cost.
- ▶ **Upper bound:** continuity of V_N on the closed set $\mathcal{Z}_N$ (contains the origin) gives $V_N(z) \leq \alpha_2(|z|)$.
- ▶ **Descent:** the warm-start inequality $V_N(z^+) \leq V_N(x, \tilde{u}) - \ell(x, \pi(x, \tilde{u}))$ with the linking lemma gives $V_N(z^+) - V_N(z) \leq -\alpha_3(|z|)$.

Practical implication: Suboptimal MPC

- ▶ Real-time implementation of MPC with early termination
- ▶ run only a few iterations of the optimization per time step (e.g., large tolerances)
- ▶ run other approximation schemes, e.g., real-time iteration (RTI) scheme.