

Lecture 6 - Economic Model Predictive Control

Dinesh Krishnamoorthy

Department of Engineering Cybernetics, NTNU

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Nonlinear system - so far!

- ▶ Consider a discrete time nonlinear time-invariant system described by the nonlinear difference equation

$$x_{k+1} = f(x_k, u_k)$$

Without loss of generality, we can assume $(0, 0)$ is the desired equilibrium with $0 = f(0, 0)$.

- ▶ System is subjected state and input constraints

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}$$

- ▶ Assume full state measured
- ~~▶ Objective is to drive the system to its origin~~
- ▶ Objective is to minimize some economic performance

Economic MPC

$$\begin{aligned} V_N(x(t)) = \min & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\ \text{s.t. } & x_{k+1} = f(x_k + u_k), & k = 0, 1, \dots, N-1 \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, & k = 0, 1, \dots, N-1 \\ & x_0 = x(t) \\ & x_N \in \mathcal{X}_f \end{aligned}$$

- ▶ MPC control law $\pi(x(t)) = u_0^*$
- ▶ Economic MPC: When $\ell(x, u)$ is **NOT** positive definite.
- ▶ General measure of economic performance often is not positive definite, e.g. prices of raw materials, utilities, value of products

Traditional Two layer paradigm

- ▶ Economic Steady-state optimization (upper layer)

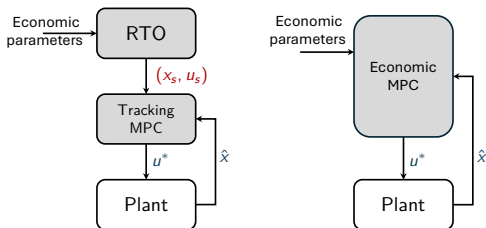
$$\begin{aligned}(\mathbf{x}_s, \mathbf{u}_s) &= \arg \min_{(\mathbf{x}, \mathbf{u})} \ell(\mathbf{x}, \mathbf{u}) \\ \text{s.t. } \mathbf{x} &= f(\mathbf{x}, \mathbf{u}) \\ \mathbf{x} &\in \mathcal{X}, \quad \mathbf{u} \in \mathcal{U}\end{aligned}$$

- ▶ Tracking MPC (lower layer)

$$\begin{aligned}\min \quad & \sum_{k=0}^{N-1} \left[(\mathbf{x}_k - \mathbf{x}_s)^\top Q (\mathbf{x}_k - \mathbf{x}_s) + (\mathbf{u}_k - \mathbf{u}_s)^\top R (\mathbf{u}_k - \mathbf{u}_s) \right] + V_f(\mathbf{x}_N - \mathbf{x}_s) \\ \text{s.t. } \quad & \mathbf{x}_{k+1} = f(\mathbf{x}_k + \mathbf{u}_k), \quad k = 0, 1, \dots, N-1 \\ & \mathbf{x}_k \in \mathcal{X}, \quad \mathbf{u}_k \in \mathcal{U}, \quad k = 0, 1, \dots, N-1 \\ & \mathbf{x}_0 = \mathbf{x}(t) \\ & \mathbf{x}_N \in \mathcal{X}_f\end{aligned}$$

Economic MPC

Economic MPC collapses the two layers by putting economic cost directly into the dynamic problem



$$V_N(x(t)) = \min \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N)$$

$$\text{s.t. } x_{k+1} = f(x_k + u_k), \quad k = 0, 1, \dots, N-1$$

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k = 0, 1, \dots, N-1$$

$$x_0 = x(t)$$

$$x_N \in \mathcal{X}_f$$

Economic MPC: when ℓ is not positive definite – Energy

Battery / grid arbitrage. State x = state of charge, input u = grid power ($u > 0$ buy/charge, $u < 0$ sell/discharge)

$$x^+ = x + \eta u, \quad 0 \leq x \leq x_{\max}, \quad |u| \leq u_{\max}$$

Economic stage cost: $\ell_k(x, u) = p_k u$ (price $\times$ power bought)

Why ℓ is not positive definite:

- **Sign-indefinite:** discharging at a positive price ($u < 0$, $p_k > 0$) gives $\ell < 0$ – you *earn* money.
- **Linear in u :** no bowl-shaped minimum; the optimum sits at a boundary $u = \pm u_{\max}$.
- **Time-varying p_k :** the economic optimum is to charge when cheap, discharge when expensive – a *cycle*, not a steady state.

Tracking analogue $\ell = (x - x_{\text{ref}})^2$ is positive definite – but ignores the price entirely.

Economic MPC: when ℓ is not positive definite – Finance

Holding a dividend-paying asset. State x = holdings (shares), input u = shares bought/sold

$$x^+ = x + u, \quad |x| \leq \bar{x} \quad (\text{position limit})$$

Economic stage cost: $\ell_k(x, u) = -d x + c |u|$ (–income + transaction fee)

Why ℓ is not positive definite:

- **Negative-valued:** while holding the asset ($x > 0$, $d > 0$), $\ell < 0$ – the “cost” is income.
- **Monotone in x :** no interior minimum; the optimum is pushed to the position limit $x = \bar{x}$, where $\ell \neq 0$.
- With a time-varying return $-r_k x$: buy-low / sell-high *cycling* – the optimum is not a fixed point.

Tracking would drive x to a fixed setpoint – but the goal here is to *maximise wealth*, not to sit still.

Economic MPC: when ℓ is not positive definite – Chemical process

Reactor profit. State x = concentration / temperature, input u = feed rate

$$x^+ = f(x, u), \quad x \in \mathcal{X}, \quad u \in \mathcal{U}$$

Economic stage cost:
$$\ell(x, u) = \underbrace{c_F u}_{\text{feed} + \text{utility}} - \underbrace{c_P g(x, u)}_{\text{product revenue}}$$

Why ℓ is not positive definite:

- **Negative when profitable:** whenever revenue beats cost, $\ell < 0$.
- **Optimum at a constraint boundary:** maximum profit = maximum throughput / conversion (max feed or temperature), where $\ell \neq 0$.
- **Possibly periodic:** forced oscillation of feed / temperature can beat any steady operation – the optimum may be a *cycle*.

Example

Consider a linear system:

$$x_{k+1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.75 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Constraint set

$$\mathcal{X} := [-10, 10] \times [-10, 10], \quad \mathcal{U} = [-1, 1]$$

Tracking stage cost

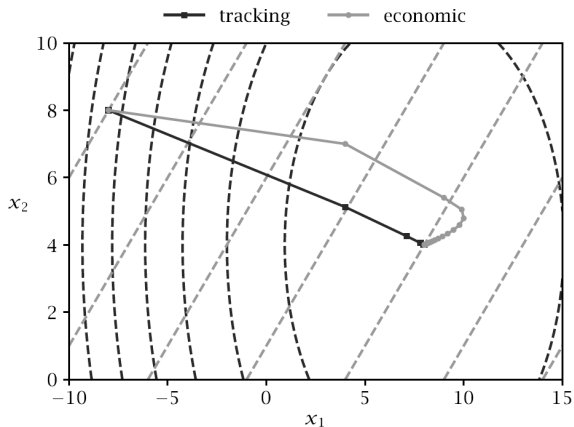
$$\ell_{\text{track}}(x, u) = \|x - x_s\|_{10I} + \|u - u_s\|_I$$

Economic stage cost

$$\ell(x, u) = [-2 \quad 2]x - 10u$$

Example

Closed-loop trajectory



Theoretical Implications of Economic MPC

Optimality?

- ▶ Closed-loop trajectories don't necessarily head directly towards the steady-state equilibrium.

Stability?

- ▶ So far, stability analysis of NMPC required the assumption

$$\ell(x, u) \geq \alpha_1(|x|) \text{ with } \alpha_1 \in \mathcal{K}_\infty$$

- ▶ and we were able to show that

$$V_N(x^+) - V_N(x) \leq -\ell(x, \pi(x)) \leq -\alpha_1(|x|)$$

- ▶ The change that $\ell(x, u)$ is not positive definite, breaks the theoretical guarantees!

Economic MPC design

Assume:

1. The functions $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n$, $\ell : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$ are continuous.
2. The terminal set is a single point $\mathcal{X}_f = \{x_s\}$. Consequently, the terminal cost $V_f(\cdot) = 0$.
3. There exists at least one feasible steady-state, i.e.,

$$(x_s, u_s) \in \mathcal{X} \times \mathcal{U} \text{ satisfying } x_s = f(x_s, u_s)$$

4. $\ell(x, u)$ is **NOT** positive definite, but lower bounded.
5. The set $\mathcal{Z} := \mathcal{X} \times \mathcal{U}$ is closed. $\mathcal{U}$ is compact and uniformly bounded in $\mathcal{X}$.

Result 1: Average performance no worse than steady-state

Average closed-loop performance is better than the best steady-state w.r.t. the economic cost $\ell(x, u)$.

For every $x \in \mathcal{X}_{feas}$, the closed-loop satisfies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{N-1} \ell(x_k, u_k) \leq \ell(x_s, u_s)$$

Economic MPC – Result 1: Asymptotic average performance

Setting: economic MPC with terminal equality constraint $\mathcal{X}_f = \{x_s\}$, $V_f = 0$, where (x_s, u_s) is the *best economic steady state*:

$$x_s = f(x_s, u_s), \quad (x_s, u_s) = \arg \min_{(x, u) \in \mathcal{Z}} \{\ell(x, u) : x = f(x, u)\}.$$

Result 1. For every $x_0 \in \mathcal{X}_{feas}$, the closed loop $x_{k+1} = f(x_k, \pi(x_k))$ satisfies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ell(x_k, u_k) \leq \ell(x_s, u_s).$$

The average closed-loop economic cost is **no worse** than the best steady-state cost.

Step 1: A one-step inequality from the terminal constraint

At state x , the optimal sequence has cost $V_N(x)$ and ends at $x_N^* = x_s$.

Successor state: $x^+ = f(x, \pi(x))$.

Candidate sequence for x^+ : shift and append u_s (feasible since $f(x_s, u_s) = x_s$):

$$\tilde{u} = \{u_1^*, \dots, u_{N-1}^*, u_s\}, \quad \text{ending again at } x_s.$$

Its cost:

$$\begin{aligned} V_N(x^+, \tilde{u}) &= \sum_{k=1}^{N-1} \ell(x_k^*, u_k^*) + \ell(x_s, u_s) \\ &= V_N(x) - \ell(x, \pi(x)) + \ell(x_s, u_s). \end{aligned}$$

Since $V_N(x^+) \leq V_N(x^+, \tilde{u})$ by optimality:

$$V_N(x^+) \leq V_N(x) - \ell(x, \pi(x)) + \ell(x_s, u_s)$$

Step 2: Telescoping sum

Rearrange the one-step inequality:

$$\ell(x_k, u_k) \leq V_N(x_k) - V_N(x_{k+1}) + \ell(x_s, u_s).$$

Sum over $k = 0, \dots, t-1$; the value-function terms telescope:

$$\begin{aligned} \sum_{k=0}^{t-1} \ell(x_k, u_k) &\leq \underbrace{\sum_{k=0}^{t-1} [V_N(x_k) - V_N(x_{k+1})]}_{= V_N(x_0) - V_N(x_t)} + t \ell(x_s, u_s) \\ &= V_N(x_0) - V_N(x_t) + t \ell(x_s, u_s). \end{aligned}$$

Step 3: Divide by t and take the limit

Divide by t :

$$\frac{1}{t} \sum_{k=0}^{t-1} \ell(x_k, u_k) \leq \frac{V_N(x_0) - V_N(x_t)}{t} + \ell(x_s, u_s).$$

- ▶ Since ℓ is **lower bounded** on $\mathcal{Z}$, so is V_N : $V_N(x_t) \geq M$ for all t .
- ▶ Hence $\frac{V_N(x_0) - V_N(x_t)}{t} \leq \frac{V_N(x_0) - M}{t} \rightarrow 0$ as $t \rightarrow \infty$.

Taking $\limsup_{t \rightarrow \infty}$:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ell(x_k, u_k) \leq \ell(x_s, u_s)$$



Note: this does *not* imply convergence to x_s – only that the average is no worse. Cycling can do strictly better.

What about stability?

- ▶ When we looked at the closed-loop decrease, we had

$$V(x^+) - V_N(x) \leq \ell(x_s, u_s) - \ell(x, \pi(x))$$

- ▶ The r.h.s. is sign indeterminate $\Rightarrow$ we cannot show the descent property
- ▶ Hence $V_N(x)$ can no longer serve as the candidate Lyapunov function.

We need a new tool to analyze stability: **Dissipativity**

Result 1 gives performance, not stability

- ▶ Result 1 says the average cost is no worse than $\ell(x_s, u_s)$ – but this is **not** a stability statement.
- ▶ The one-step inequality carried a **sign-indefinite** term:

$$V_N(x^+) - V_N(x) \leq -(\ell(x, \pi(x)) - \ell(x_s, u_s)),$$

which need not be negative $\Rightarrow V_N$ is **not** a Lyapunov function.

- ▶ For some systems **cycling** beats the steady state, and economic MPC *should* not converge.

When does economic MPC actually stabilize the steady state x_s ?

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When does economic MPC actually stabilize the steady state x_s ?

Answer: the notion of **dissipativity**.

Dissipativity

Think of the system as a **battery**:

- ▶ **Storage function** $\lambda(x)$ = energy currently stored (state of charge).
- ▶ **Supply rate** $s(x, u)$ = energy supplied over one step (charging power in).
- ▶ **Dissipation** = energy lost to internal losses (heat, leakage).

$$\text{Energy balance:} \quad \underbrace{\lambda(x^+) - \lambda(x)}_{\text{change in stored energy}} \leq \underbrace{s(x, u)}_{\text{energy supplied}}$$

- You can **never store more than you supply** – the deficit is dissipated (lost).
- No losses $\Rightarrow$ equality (energy conserved); with losses $\Rightarrow$ strict inequality.

Dissipativity: formal definition

Definition (dissipativity). The system $x^+ = f(x, u)$ is **dissipative** with respect to a supply rate $s(x, u)$ if there exists a **storage function** $\lambda : \mathcal{X} \rightarrow \mathbb{R}$ such that

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u), \quad \forall (x, u) \in \mathcal{Z}.$$

Strict dissipativity. In addition, there exists $\rho \in \mathcal{K}_\infty$ with

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) - \rho(|x - x_s|), \quad \forall (x, u) \in \mathcal{Z}.$$

For **economic MPC**, take the supply rate as the **excess cost** over the best steady state:

$$s(x, u) = \ell(x, u) - \ell(x_s, u_s).$$

Battery view: $\rho(|x - x_s|)$ is the **guaranteed loss** incurred whenever the state is away from x_s .

Result 2: Stability of Economic MPC

Assume

- ▶ The system $x^+ = f(x, u)$ is strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(x_s, u_s)$

Given these assumptions, x_s is asymptotically stable in $\mathcal{X}_{feas}$ for the closed-loop system $x^+ = f(x, \pi(x))$

- ▶ Proving this relies on finding a suitable candidate Lyapunov function.
- ▶ But $V_N(x)$ cannot serve this purpose!

Result 2: Stability of Economic MPC

- Define the **rotated stage cost**

$$\tilde{\ell}(x, u) = \underbrace{\ell(x, u) - \ell(x_s, u_s)}_{s(x, u)} + \underbrace{\lambda(x) - \lambda(f(x, u))}_{-\Delta \text{ storage}}.$$

- Strict dissipativity gives exactly what we need:

$$\begin{aligned}\tilde{\ell}(x, u) &\geq \rho(|x - x_s|) \quad (\text{positive definite about } x_s) \\ \tilde{\ell}(x_s, u_s) &= 0\end{aligned}$$

- New candidate Lyapunov function constructed in terms of the rotated cost

$$\tilde{V}_N(x) := \sum_{k=0}^{N-1} \tilde{\ell}(x(k), u(k))$$

Question: But do they give the same control law $\pi(x)$? - **Yes!**

Result 2: Stability of Economic MPC

$\tilde{V}_N$ and V_N differ only by constants $\Rightarrow$ same optimal control law.

$$\begin{aligned}\tilde{V}_N &= \sum_{k=0}^{N-1} \tilde{\ell}(x(k), u(k)) \\ &= \sum_{k=0}^{N-1} \ell(x(k), u(k)) - N\ell(x_s, u_s) + \lambda(x) - \lambda(x_s) \\ &= V_N - N\ell(x_s, u_s) + \lambda(x) - \lambda(x_s)\end{aligned}$$

Since the rotated stage cost is non-negative, $\tilde{V}_N$ being a sum of the stage costs,

- $\tilde{V}_N$ is a valid **Lyapunov function** $\Rightarrow x_s$ is asymptotically stable.