

Lecture 7 - Inherent Robustness of Nominal MPC

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So far, so good!

- So far, we considered the nominal MPC problem designed for the **disturbance-free** model

$$x^+ = f(x, u).$$

and showed that **in theory**, the closed loop system $x^+ = f(x, \pi(x))$ is stable.

- But **in practice**, the real system is subject to disturbances

$$x^+ = f(x, \pi(x + e)) + d$$

where d is the process disturbance, and e is the measurement noise.

- what use is a controller if it does not ensure the desired behaviour under small disturbances?

Does nominal stability survive a small disturbance?

Input-to-State Stability (ISS)

Consider a system

$$x^+ = f(x, w)$$

with input w .

Definition (ISS). The system $x^+ = f(x, w)$ is **input-to-state stable** on a set if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all x_0 and a bounded disturbance sequences w ,

$$|x(k)| \leq \beta(|x_0|, k) + \gamma(\|w\|_{0:k-1}), \quad \forall k \geq 0.$$

where $\|w\|_{a:b} := \max_{j \in \mathbb{I}[a,b]} |w(j)|$

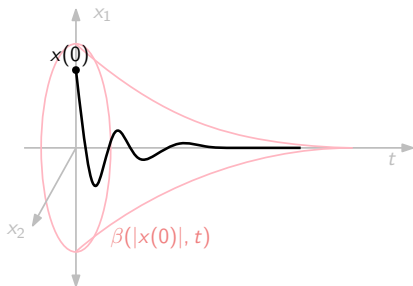
- ▶ **Transient term** $\beta(|x_0|, k)$: decays to zero (effect of the initial state).
- ▶ **Gain term** $\gamma(\|w\|)$: bounded by the disturbance size.

Small $w \Rightarrow$ small ultimate bound; $w = 0 \Rightarrow$ nominal asymptotic stability is recovered.

Global asymptotic stability with 0-input

A system $\dot{x} = f(x, 0)$ is globally asymptotically stable at zero if there exists β of class $\mathcal{KL}$ for all initial values $x(0) \in \mathbb{R}^n$ such that

$$|x(k)| \leq \beta(|x_0|, k), \quad \forall k \geq 0.$$



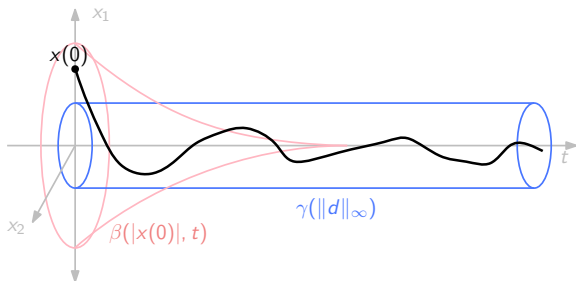
The $\mathcal{KL}$ function characterizes transient phenomenon that is proportional to $x(0)$, and fades to zero as $k \rightarrow \infty$.

Input-to-state Stability

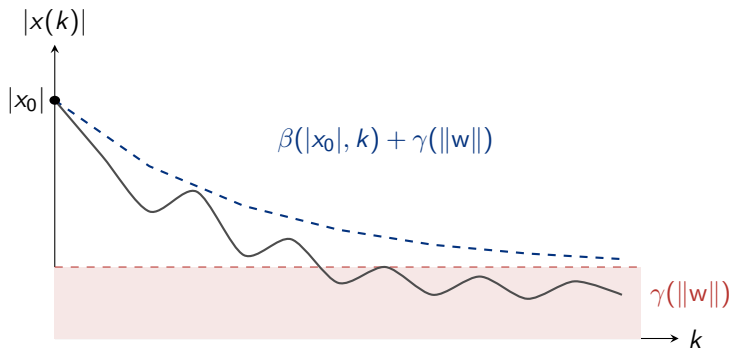
ISS stability

A system is ISS stable if there exists β and γ of class $\mathcal{K}_\infty$, such that for all $x_0 \in \mathbb{R}^n$, and all measurable locally bounded inputs w , the following holds:

$$|x(k)| \leq \beta(|x_0|, k) + \gamma(\|w\|), \quad \forall k \geq 0.$$



ISS: the tube picture

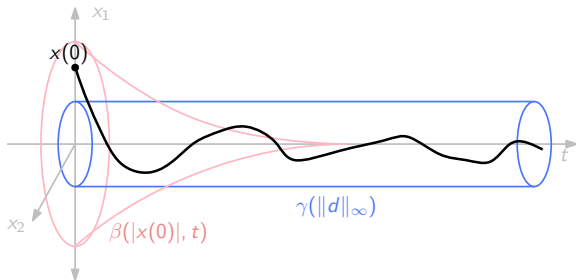


The state stays under a **decaying envelope** and settles into the **$\gamma(\|w\|)$ band** (the RPI set).

Input-to-state Stability

$$|x(t)| \leq \max[\beta(|x(0)|, t), \gamma(\|d\|_\infty)]$$

- ▶ Bounded-input bounded-state: if $\|w\|$ is bounded, then $x(k)$ is also bounded.
- ▶ For $w = 0$, this boils down to asymptotic stability
- ▶ Converging-input converging-state: If $w(k) \rightarrow 0$ as $k \rightarrow \infty$, then $x(k) \rightarrow 0$ asymptotically.



Robust positive invariance

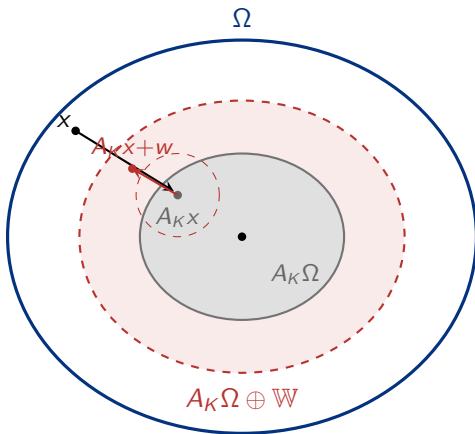
Under persistent disturbances the state cannot reach the exact origin – it must converge to and stay in a set.

Definition (RPI). A set Ω is **robust positively invariant** for $x^+ = f(x, w)$, if there exists a $\delta > 0$ such that for all disturbance sequences satisfying $\|w\| \leq \delta$

$$x \in \Omega \Rightarrow f(x, w) \in \Omega, \quad \forall x \in \mathcal{X}_{feas}$$

- ▶ Once inside Ω , **no** admissible disturbance can push the state out.
- ▶ Linear case: $A_K \Omega \oplus \mathbb{W} \subseteq \Omega$ (map the set, inflate by $\mathbb{W}$, still fits inside).
- ▶ The ISS gain sets the **size** of Ω – proportional to the disturbance magnitude.

Robust positive invariance: the picture



Ω is RPI $\iff A_K \Omega \oplus \mathbb{W} \subseteq \Omega$ (map the set, inflate by $\mathbb{W}$, still fits inside)

ISS-Lyapunov function

V is an **ISS-Lyapunov function** if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\begin{aligned}\alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ V(f(x, w)) - V(x) &\leq -\alpha_3(|x|) + \sigma(\|w\|).\end{aligned}$$

Theorem. If the closed loop admits an ISS-Lyapunov function, then it is ISS.

Strategy: show that the *nominal* value function V_N is an ISS-Lyapunov function for the *perturbed* closed loop, with $w = [d, e]$ as the input.

Inherent robustness

- ▶ measured state: $\hat{x} = x + e$
- ▶ compute policy using the measured state: $u^* = \pi(\hat{x})$
- ▶ successor state: $x^+ \in f(x, \pi(\hat{x})) = f(\hat{x} - e, \pi(\hat{x})) + d$
- ▶ Measured successor state: $\hat{x}^+ \in f(\hat{x} - e, \pi(\hat{x})) + d + e^+$

Since f is continuous,

$$\begin{aligned} |\hat{x}^+ - f(\hat{x}, \pi(\hat{x}))| &= |f(\hat{x} - e, \pi(\hat{x})) + d + e^+ - f(\hat{x}, \pi(\hat{x}))| \\ &\leq |f(\hat{x} - e, \pi(\hat{x})) - f(\hat{x}, \pi(\hat{x}))| + |d| + |e^+| \\ &\leq \sigma_f(|e|) + |d| + |e^+| \leq \sigma_1|w| \end{aligned}$$

Inherent robustness

Since V_N is continuous

$$\begin{aligned} |V_N(\hat{x}^+, \tilde{\mathbf{u}}) - V_N(f(\hat{x}, \pi(\hat{x})), \tilde{\mathbf{u}})| &\leq \sigma_V(|\hat{x}^+ - f(\hat{x}, \pi(\hat{x}))|) \\ &\leq \sigma_V(\sigma_1|w|) := \sigma(|w|) \\ \Rightarrow V_N(\hat{x}^+, \tilde{\mathbf{u}}) &\leq V_N(f(\hat{x}, \pi(\hat{x})), \tilde{\mathbf{u}}) + \sigma(|w|) \end{aligned}$$

From the nominal stability proof, we have

$$\begin{aligned} V_N(f(\hat{x}, \pi(\hat{x})), \tilde{\mathbf{u}}) &\leq V_N(\hat{x}) - \underbrace{\ell(\hat{x}, \pi(\hat{x}))}_{\geq \alpha_1(|\hat{x}|)} \\ V_N(\hat{x}^+, \tilde{\mathbf{u}}) &\leq V_N(\hat{x}) - \alpha_1(|\hat{x}|) + \sigma(|w|) \\ \Rightarrow V_N(\hat{x}^+, \tilde{\mathbf{u}}) - V_N(\hat{x}) &\leq -\alpha_1(|\hat{x}|) + \sigma(|w|) \end{aligned}$$

$$V_N(x^+) - V_N(x) \leq -\alpha_3(|x|) + \sigma(|w|)$$

Together with $\alpha_1(|x|) \leq V_N(x) \leq \alpha_2(|x|)$, this is exactly the **ISS-Lyapunov** condition.

Inherent robustness: conclusion

- ▶ V_N is an **ISS-Lyapunov function** for the perturbed closed loop $\Rightarrow$ the closed loop is **ISS**:

$$|x(k)| \leq \beta(|x_0|, k) + \gamma(\|w\|).$$

- ▶ The state converges to a **robust positively invariant** set Ω about the origin, with size proportional to $\|w\|$.

Inherent robustness. A *nominal* MPC with a *continuous* value function is automatically ISS – small disturbances cause only small, bounded deviations. ■

Caveat: state / terminal constraints can make V_N discontinuous $\Rightarrow$ possibly **zero** robustness (no constraint guarantees).