TTK4115 - Linear System theory Fall 2025 DINESH.K

Continous time signal

→ A signal defined at every instant of time.

Continous time system

→ A system that accepts a CT signal as input and generates a CT signal as output.

Linear system

→ A system is linear if it satisfies

↳ additivity : $f(x_1 + x_2) = f(x_1) + f(x_2)$ } superposition

↳ Homogeneity : $f(\alpha x) = \alpha f(x)$ }

A system is linear if it satisfies the superposition principle

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

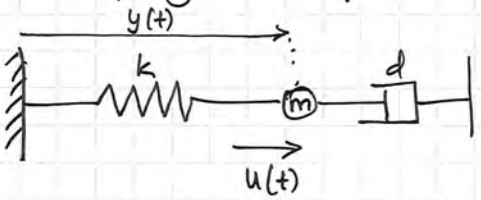
X How to represent a linear system?
- → State-space
- → transfer function
- → input / output description.

Causal system / Causality :

Current output depends on current & past input but not on future input

All physical systems are causal.

Mass - Spring damper



input $u(t)$ = force

output $y(t)$ = displacement

Second order dynamics

$$m\ddot{y}(t) + d\,\dot{y}(t) + k\,y(t) = u(t)$$

States $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$ position / velocity

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad y(t) = C\,x(t) + D\,u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$\ddot{y}(t) = \frac{1}{m}u(t) - \frac{k}{m}y(t) - \frac{d}{m}\dot{y}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

$$A := \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix} \qquad B := \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \qquad C := \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D := 0$$

State-space → transfer function

$$x(t) \Leftrightarrow \hat{x}(s)$$

$$\dot{x}(t) \Leftrightarrow s\hat{x}(s) - x_0$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

↓ Laplace transform

$$sX(s) - X_0 = AX(s) + BU(s)$$

$$(sI - A)X(s) = X_0 + BU(s)$$

$$X(s) = (sI - A)^{-1}[X_0 + BU(s)]$$

$$Y(s) = C(sI - A)^{-1}[X_0 + BU(s)] + DU(s)$$

$$Y(s) = \underbrace{C(sI - A)^{-1}X_0}_{\text{Zero input}} + \underbrace{[C(sI - A)^{-1}B + D]U(s)}_{\text{Zero state}}$$

⇓

$G(s)$ Transfer matrix

Mass-Spring damper system

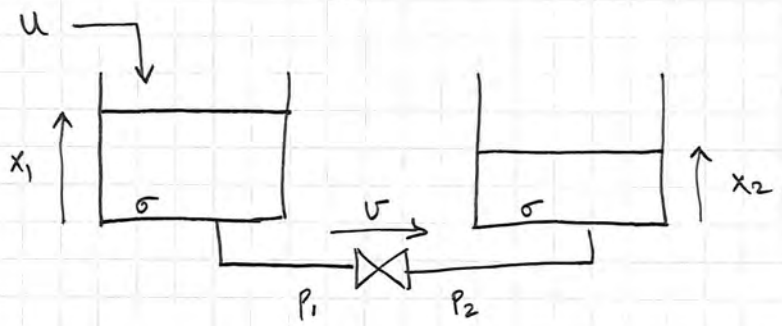
$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} s & -1 \\ k/m & s + \frac{d}{m} \end{bmatrix}^{-1}}_{\text{''}} \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$= \frac{1}{s(s + \frac{d}{m}) + \frac{k}{m}} \begin{bmatrix} s + \frac{d}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \Rightarrow \frac{1}{ms^2 + ds + k} \begin{bmatrix} ms + d & m \\ -k & ms \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{ms+d}{ms^2+ds+k} & \frac{m}{ms^2+ds+k} \\ \frac{-k}{ms^2+ds+k} & \frac{ms}{ms^2+ds+k} \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \Rightarrow \frac{1}{ms^2 + ds + k}$$

Tank system



input - vol. flow into tank 1

output = $(x_1 + x_2)/2$ avg. tank level

$$P_1 = \rho g x_1 \qquad P_2 = \rho g x_2$$

$$v = k(P_1 - P_2) = k\rho g(x_1 - x_2)$$

$$\sigma \dot{x}_1 = u - v = u - k\rho g(x_1 - x_2) = u - k\rho g x_1 + k\rho g x_2$$

$$\sigma \dot{x}_2 = v = k\rho g(x_1 - x_2) = k\rho g x_1 - k\rho g x_2$$

States: level in the tanks $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k\rho g}{\sigma} & \frac{k\rho g}{\sigma} \\ \frac{k\rho g}{\sigma} & -\frac{k\rho g}{\sigma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/\sigma \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{bmatrix} \qquad B = \begin{bmatrix} 1/\sigma \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \qquad D = 0$$

$$\alpha = \frac{k\rho g}{\sigma}$$

Define alternative states

$\bar{x}_1 = \frac{1}{2}(x_1 + x_2)$ (average)

$\bar{x}_2 = (x_1 - x_2)$ difference

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix}}_{:= T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transformation matrix

New state space model:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \bar{A} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \bar{B} u$$

Equivalence transformation

$$y = \bar{C} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \bar{D} u$$

$\bar{A} = TAT^{-1}$ $\quad \bar{B} = TB$

$\bar{C} = CT^{-1}$ $\quad \bar{D} = D$ (direct transmission)

$$\begin{aligned} \bar{G}(s) &= \bar{C}(s\bar{I} - \bar{A})^{-1}\bar{B} + \bar{D} \\ &= CT^{-1}\left(T(sI - A)T^{-1}\right)^{-1} TB + D \\ &= CT^{-1}T\,(sI - A)^{-1}\,T^{-1}TB + D \\ &= C(sI - A)^{-1}B + D = G(s) \end{aligned}$$

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Compute e^{At}

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$k = 10$, Compute A^{10}

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda - 2 & 4 \\ 0 & \lambda - 1 \end{bmatrix}$$

$$(\lambda - 2)(\lambda - 1) - 0 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

Casey - Hamilton theorem

$$A^2 - 3A + 2 = 0$$

$$A^2 = 3A - 2$$

$$A^3 = AA^2 = A(3A - 2) = 3A^2 - 2A$$

$$= 3(3A - 2) - 2A$$

$$\Rightarrow 9A - 6 - 2A$$

$$\Rightarrow 7A - 6$$

$$A^4 = AA^3 = A(7A - 6) = 7A^2 - 6A$$

$$7(3A - 2) - 6A$$

$$\Rightarrow 21A - 14 - 6A$$

$$\Rightarrow \underline{15A - 14}$$

Equivalence transforms = Similarity

$$\bar{x} = \bar{T}x \qquad A = \begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 2\alpha \end{bmatrix}$$

$$G(s) = C\left((sI - A)^{-1}B + D\right)$$

$$\frac{4}{6} \quad \frac{2}{3} \quad \frac{8}{12}$$

↳ canonical

Eigenvalues and vectors

$$Aq = \lambda q$$

$$(A - \lambda I)q = 0$$

($A - \lambda I$: not invertable)

$$\det(A - \lambda I) = 0$$

$$\det(\lambda I - A) = 0$$

EX//

$$A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \Rightarrow \det\left(\begin{bmatrix} \lambda+2 & -2 \\ 2 & \lambda-3 \end{bmatrix}\right)$$

$$= (\lambda+2)(\lambda-3) - (-2)2 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = -1$$

$$\lambda_1 : (A - 2I)q_1 = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} q_1 = 0 \Rightarrow q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 : (A - (-1)I)q_2 = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} q_2 = 0 \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Diagonalization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$X = T^{-1} \bar{X}$$

$$X(t) = \alpha_1(t) q_1 + \alpha_2(t) q_2 + \cdots + \alpha_n(t) q_n$$

$$= \underbrace{[q_1 \; q_2 \cdots q_n]}_{Q} \underbrace{\begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix}}_{\alpha(t)} = Q\alpha(t)$$

$$X(t) = Q\alpha(t) \qquad \alpha = \bar{x}$$

$$X(t) = Q\bar{x}(t) \Rightarrow \dot{x}(t) = Q\dot{\bar{x}}(t)$$

$\dot{x}(t) = Ax(t) \longrightarrow n_0$ inputs

$$Q\dot{\bar{x}}(t) = AQ\bar{x}(t) = A[q_1 \cdots q_n]\bar{x}(t)$$

$$= [Aq_1 \cdots Aq_n]\bar{x}(t)$$

$$= [\lambda_1 q \cdots \lambda_n q_n]\bar{x}(t)$$

$$= [q_1 \cdots q_n] \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda} \bar{x}(t)$$

$$Q\dot{\bar{x}}(t) = Q\Lambda\bar{x}(t)$$

$$\boxed{T = Q^{-1}}$$

$$\dot{\bar{x}}(t) = \Lambda\bar{x}(t) =$$

$\in X$ //

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda & -2 \\ 0 & -1 & \lambda-1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -2 \\ -1 & \lambda-1 \end{vmatrix}$$

$$= \lambda(\lambda^2 - \lambda - 2) = \lambda(\lambda-2)(\lambda+1)$$

$$\rightarrow \lambda_1 = 0, \; \lambda_2 = 2, \; \lambda_3 = -1$$

$\rightarrow \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -1$

$$q_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

Diagonalization → Key takeaways

$$\bar{A} = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad T = Q^{-1}, \quad Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$X = T^{-1}\bar{X}, \quad \bar{X} = TX, \quad T = Q^{-1}$$

states pace

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$X = Q\bar{X}$$

$$Q\dot{\bar{x}} = AQ\bar{x} + Bu \Rightarrow \dot{\bar{x}} = \underbrace{Q^{-1}AQ}_{\Lambda}\bar{x} + \underbrace{Q^{-1}B}_{\bar{B}}u$$

$$y = \underbrace{CQ}_{\bar{C}}\bar{x} + \underbrace{D}_{\bar{D}}u$$

A must have n distinct λ for always being diagonalizable!

Jordan form

$$\Lambda = Q^{-1}AQ \Rightarrow Q\Lambda = AQ$$

$$\Lambda = Q^{-1} A Q \Rightarrow Q\Lambda = AQ$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Ex //

$$A = \begin{bmatrix} 4 & -1 \\ 4 & 8 \end{bmatrix} \Rightarrow |\lambda I - A| = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2$$

$$\Rightarrow \lambda_{1,2} = 6$$

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} q = 0 \Rightarrow q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Jordan block

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Instead of $AQ = Q\Lambda$ we want to find

$AQ = Q$) Now $Q = [V_1 \; V_2]$

$$A[V_1 \; V_2] = [V_1 \; V_2] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow AV_1 = \lambda V_1$$

↳ Eigenvector

$$AV_2 = V_1 + \lambda V_2 \Rightarrow (A - \lambda I) V_2 = V_1$$

↳ generalized eigenvector

Ex cont //

$$A = \begin{bmatrix} 4 & -1 \\ 4 & 8 \end{bmatrix}, \quad \lambda_{1,2} = 6, \quad q_1 = V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(A - \lambda I) V_2 = V_1 \Rightarrow (A - 6I) V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

↳ $\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} -1 \\ \end{bmatrix}$

↳ $\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \quad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Ex // Exam 2020

$\dot{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \;,\; y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$

① Find eigenvalues

See from A that $\lambda_{1,2} = 1$

② Find eig vec

$(A - 1 \cdot I) q_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

③ Find generalized eigenvector

(→ gen. eig vec)

$(A - \lambda I) v = q \Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$Q = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}, \quad T = Q^{-1} \quad Q^{-1} = \frac{1}{0 - \frac{1}{2}} \begin{bmatrix} 0 & -1/2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

TTK4115 → L2, part 2

Key take aways

① Eq. transform

$\bar{x} = Tx$, if T is invertable

$x = T^{-1}\bar{x}$

can go from:

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$\longrightarrow$

$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$

$y = \bar{C}\bar{x} + \bar{D}u$

$\bar{A} = TAT^{-1}$

$\bar{B} = TB$

$\bar{C} = CT^{-1}$

$\bar{D} = D$

② Canonical forms

a) Diagonalization

Find λ and Q and use $T = Q^{-1}$

- Need to have n linearly indep. q !

b) Jordan

Find λ find eigenvecs and gen. eigvec.

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

c) Modal form

- When we get complex conjugated eigenvalues
- Applied after diagonalization
- we always use $M = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$

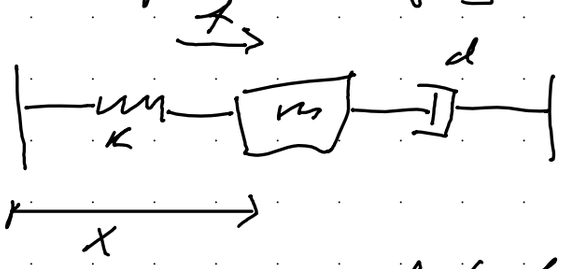
$$M^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

Why?

- Physical inputs/outputs are real
- Complex stuff is hard....
- Real values more computationally efficient!

Modal forms

complex conjugate: $\alpha \pm j\beta$



$m\ddot{x} + d\dot{x} + kx = F$

$x_1 = x$, $x_2 = \dot{x}$

- We use modal form to "remove" complex values

⇒ Easier for us! :)

Always use $T = M^{-1}$, $M = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix}$$

$$M^{-1}\Lambda M = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \alpha+\beta i & 0 \\ 0 & \alpha-\beta i \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+\beta i & \alpha-\beta i \\ \alpha i-\beta & -\alpha i-\beta \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha+\beta i+\alpha-\beta i}{2} & \frac{-\alpha i+\beta+\alpha i+\beta}{2} \\ \frac{\alpha i-\beta-\alpha i-\beta}{2} & \frac{\alpha+\beta i+\alpha-\beta i}{2} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Block diagonal matrix

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & 1 & \\ & & \lambda_2 & \\ & 0 & & \lambda_3 \\ & & & & \alpha & \beta \\ & & & & -\beta & \alpha \end{bmatrix}$$

A^{10} $\quad A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$

$f(\lambda) = \lambda^{10}$, what is A^{10}

$\lambda_1 = 1$.
$\lambda_2 = 2$.

eigenvalues are unique.

$h(\lambda) = \beta_0 + \beta_1 \lambda$

$f(\lambda_i) = h(\lambda_i)$

~~$\beta_0 + \beta_1 = 0$~~ $\quad$ ~~$\beta_0 = -\beta_1$~~

~~$\beta_0 + 2\beta_2 = 0$~~

$\lambda = 1 \quad f(\lambda = 1) = \lambda^{10} = 1^{10} = 1 = \beta_0 + \beta_1$

$\lambda = 2 \quad f(\lambda = 2) = 2^{10} = \beta_0 + 2\beta_1$

$$\beta_0 + \beta_1 = 1$$
$$\beta_0 + 2\beta_1 = 2^{10}$$

$\Rightarrow \quad \beta_0 = -1022, \quad \beta_1 = 1023$

Casey Hamilton theorem

$f(A) = h(A) =$ ~~$\beta_0 I$~~ $\quad \beta_0 I + \beta_1 A = A^{10}$

$$A^{10} = -1022 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1023 \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

$\underline{f(A)} = A^{15}$ $\quad A \in \mathbb{R}^{n \times n}$

$f(\lambda) = \lambda^{15}$

$f(\lambda) = h(\lambda)$

Compute eigenvalues of A

λ_1
$\vdots$
λ_n } if unique

$h(\lambda) : \beta_0 \underline{I} + \dots + \beta_{n-1} A^{n-1}$

Any matrix function $f(A)$ can be expressed as a matrix polynomial of order $n-1$

$f(\lambda_1) = h(\lambda_1) = \beta_0 + \beta_1 \lambda_1 + \dots + \beta_{n-1} \lambda_1^{n-1}$

$f(\lambda_2) = h(\lambda_2) = \dots$

$\vdots$

$f(\lambda_n) = h(\lambda_n) = \dots$

n equations, n unknowns

solve this system of linear equations to compute

$\beta_0, \dots \beta_{n-1}$

$f(A) = \beta_0 \underline{I} + \beta_1 A + \dots + \beta_{n-1} A^{n-1}$

$\lambda_1 = \lambda_2 = \lambda$

① $\Rightarrow f(\lambda) = \beta_0 + \beta_1 \lambda$

$\frac{\partial f}{\partial \lambda} = \frac{\partial h}{\partial \lambda} = \beta_1$

② $\rightarrow \beta_1 = \frac{\partial f}{\partial \lambda}(\lambda)$

→ if $f(\lambda) = h(\lambda)$

then their derivatives must also be equal

→ use this fact when we have repeated eigenvalues.

λ_1

λ_2

$\vdots$

repeated eigenvalues: λ_4, λ_4

λ_5

λ_6

$\vdots$

$f(\lambda_1) = \beta_0 + \beta_1 \lambda_1 + \dots + \beta_{n-1} \lambda_1^{n-1}$

$f(\lambda_2) = \dots$

$\vdots$

$f(\lambda_4) = \beta_0 + \beta_1 \lambda_4 + \dots + \beta_{n-1} \lambda_4^{n-1}$

$\frac{\partial f}{\partial \lambda}(\lambda_4) = \beta_1 + \dots + \beta_n \lambda_4^{n-2}$

$\dots$

$f(\lambda_5) =$

$f(\lambda_6) =$

$f(A) = \beta_0 I + \dots + \beta_n A^{n-1}$

$$f(A) = \begin{bmatrix} f(A_1) & & & \\ & f(A_2) & & \\ & & \ddots & \\ & & & f(a_3) \\ & & & & f(a_4) \end{bmatrix}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}}_{:=A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{:=B} u$$

$T = 1$ sampling time.

$$A_d = e^{AT} \qquad B_d = A^{-1}(A_d - I)B$$

Compute e^{AT} with $T = 1$

given $f(\lambda) = e^{\lambda}$, compute $f(A) = e^{A}$

→ eigenvalues $\lambda_1 = 2$; $\lambda_2 = 1$

$$h(\lambda) = \beta_0 + \beta_1 \lambda = f(\lambda) = e^{\lambda}$$

→ Compute β_0 & β_1

$$h(\lambda_1 = 2) = \beta_0 + 2\beta_1 = e^2$$

$$h(\lambda_2 = 1) = \beta_0 + \beta_1 = e = 2.718$$

$$\beta_1 = e^2 - e = 4.6708$$

$$\beta_0 = e - \beta_1 = e - e^2 + e \Rightarrow \quad 2e - e^2 = -1.9525$$

From Casey-Hamilton theorem

if $h(\lambda) = \beta_0 + \beta_1 \lambda = f(\lambda)$ holds,

then $\beta_0 I + \beta_1 A = f(A)$ also holds

$$\Rightarrow e^A = \beta_0 I + \beta_1 A$$

$$-1.9525 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4.6708 \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7.3891 & 18.683 \\ 0 & 2.718 \end{bmatrix}$$

$$\begin{bmatrix} -1.9525 & 0 \\ 0 & -1.9525 \end{bmatrix} + \begin{bmatrix} 9.3416 & 18.6832 \\ 0 & 4.6708 \end{bmatrix} \Rightarrow$$

$$A_d = \begin{bmatrix} 7.3891 & 18.6832 \\ 0 & 2.7183 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} c & -b \\ -d & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & -2 \\ 0 & 1 \end{bmatrix}$$

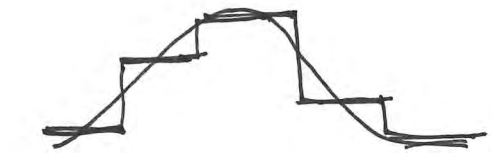
$$B_d = A^{-1}(A_d - I)B$$

$$= \begin{bmatrix} 0.5 & -2 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 7.3891 & 18.6832 \\ 0 & 2.7183 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.1946 & 5.905 \\ 0 & 1.7183 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

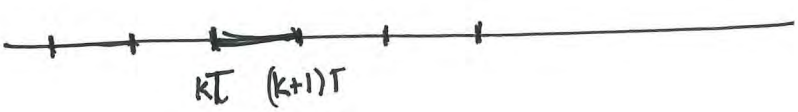
$$= \begin{bmatrix} 5.905 \\ 1.7183 \end{bmatrix}$$

check with MATLAB c2d( ) function.



piecewise constant input

$u(t) = u_k \; ; \quad kT < t < (k+1)T$





$$\dot{x} = Ax + Bu \longrightarrow \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$e^{\bar{A}T} = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix}$$

$$x_{k+1} = A_d x_k + B_d u_k$$

$$A_d = e^{AT} \qquad T = \text{sampling time}$$

$$B_d = A^{-1}\left[e^{AT} - I\right]B$$

$$= A^{-1}\left[A_d - I\right]B$$

Euler discretization

$$\dot{x} = \frac{x(k+1) - x(k)}{T} \qquad \lim_{T \to 0}$$

$$\dot{x} = Ax + Bu$$

$$\frac{x_{k+1} - x_k}{T} = Ax_k + Bu_k$$

$$\Rightarrow \quad x_{k+1} = (I + TA)x_k + TB u_k$$

$$x_{k+1} = (I + TA)\, x_k + TB\, u_k$$

$$A_d \approx I + TA$$

$$B_d \approx TB$$

Controllability:

Given a discrete time system

$$x_{k+1} = A x_k + B u_k \qquad A \in \mathbb{R}^{4\times 4}$$

starting from x_0, can we control this system to a desired x_4

$$x_1 = A x_0 + B u_0$$

$$x_2 = A x_1 = B u_1$$
$$= A(Ax_0 + Bu_0) + Bu_1 = A^2 x_0 + AB u_0 + B u_1$$

$$x_3 = A x_2 + B u_2$$
$$= A(A^2 x_0 + AB u_0 + B u_1) + B u_2 = A^3 x_0 + A^2 B u_0 + AB u_1 + B u_2$$

$$x_4 = A x_3 + B u_3$$
$$= A(A^3 x_0 + A^2 B u_0 + AB u_1 + B u_2) + B u_3$$
$$= A^4 x_0 + A^3 B u_0 + A^2 B u_1 + AB u_2 + B u_3$$

$$\Rightarrow x_4 = A^4 x_0 + \begin{bmatrix} A^3B & A^2B & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\Rightarrow x_4 - A^4 x_0 = \underbrace{\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix}$$

iff full rank, we can choose x_4 as we like with inputs

$$C := \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

For a system with $A \in \mathbb{R}^{n\times n}$

$$C := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

if $\text{rank}(C) = n \Rightarrow (A, B)$ is controllable.

Controllability Gramian

Given a system $\dot{x} = Ax + Bu$, the LTI solution is given by:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

We want to place the system at x_1 at $t = t_1$

$$x_1 = x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} B\, u(\tau)\, d\tau$$

We need an input $u(t)$ to do this.

claim: The input $u(t)$ to do this is given by:

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right]$$

Let's verify this claim

$$x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} B \underbrace{u(\tau)}_{\text{substitute here}} d\tau$$

$$x(t_1) = e^{At_1} x_0 - \int_0^{t_1} e^{A(t_1-\tau)} B \left(B^T e^{A^T(t_1-\tau)} W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right] \right) d\tau$$

$$= e^{At_1} x_0 - \underbrace{\left(\int_0^{t_1} e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} d\tau \right)}_{:= W_c(t_1)} W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right]$$

Define

$$= e^{At_1} x_0 - W_c(t_1) W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right]$$

$$= e^{At_1} x_0 - e^{At_1} x_0 + x_1$$

$$= x_1$$

$$\Rightarrow \quad x(t_1) = x_1 \quad \text{by choosing input}$$

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right]$$

where $W_c(t_1) := \int_0^{t_1} e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} d\tau$

and this input signal $u(t)$ is defined if $W_c(t_1)$ is invertible

$W_c(t_1)$ is invertible, if it has full rank.

What can we conclude from this?

if $W_c(t_1)$ has full row rank

$\Rightarrow$ $W_c(t_1)$ is invertible

if $W_c(t_1)$ is invertible,

we can compute an input $u(t) = -B^T e^{A^T(t_1 - \tau)} W_c(t_1)^{-1} \left[e^{At_1} x_0 - x_1 \right]$

that would achieve state $x = x_1$ at time $t = t_1$

$\Rightarrow$ we are able to control our system

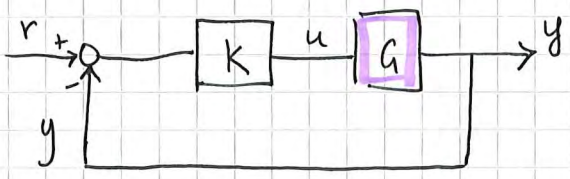
Therefore, if the controllability gramian $W_c(t_1)$ has full row rank, then the system is controllable.

Pole placement

Consider the plant $G(s) = \frac{1}{s-2}$

system is unstable (RHP pole).

Introduce feedback to control the pole



$$u = K(r-y)$$

closed loop transfer function

$$T(s) = \frac{GK}{1+GK}$$

$$\Rightarrow \quad \frac{\frac{K}{s-2}}{1+\frac{K}{s-2}} \quad \Rightarrow \quad \frac{\frac{K}{s-2}}{\frac{s-2+K}{s-2}} \quad \Rightarrow \quad \frac{K}{s-2+K}$$

what values of K will make my closed-loop transfer function stable?

$$K > 2$$

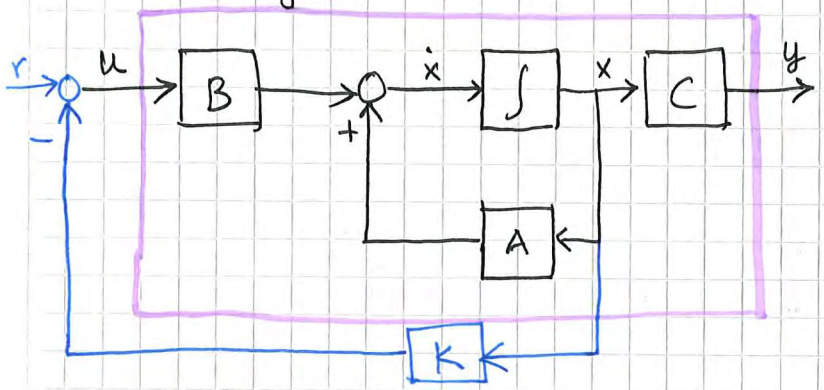
e.g. $K = 1$

$$T(s) = \frac{1}{s+1} \quad \Rightarrow \quad \text{pole in LHP} \Rightarrow \text{stable}$$

This is the idea of pole-placement in output feedback

We want the same idea in state-feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx$$



State-feedback

$$u = -Kx$$

$$\dot{x} = Ax + Bu$$
$$= Ax + B(-Kx)$$
$$= (A - BK)x$$

$$\dot{x} = (A-BK)x \quad \leftarrow \text{closed-loop system.}$$

How to choose K, such that the closed-loop system is stable?

$$u = r - Kx$$
$$\dot{x} = Ax + Br - BKx$$
$$\Rightarrow \dot{x} = (A - BK)x + Br$$

Consider the plant described by:

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

characteristic polynomial $\Delta(s) = |sI - A|$

$$\Delta(s) = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right| = \begin{vmatrix} (s-1) & -3 \\ -3 & (s-1) \end{vmatrix}$$

$$= (s-1)^2 - 9 = s^2 - 2s + 1 - 9 = s^2 - 2s - 8$$

$$= (s-4)(s+2)$$

eigenvalues are: $\underbrace{+4}_{\text{+ve real eigenvalue makes it unstable.}}$ and -2 (unstable system)

Introduce state-feedback $u = r - Kx$ $\quad K = [k_1 \quad k_2]$

$$\dot{x} = (A - BK)x + Br$$

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2] x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

Can we choose k_1 & k_2 such that $A - BK$ has negative real parts?

$$\dot{x} = \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$\Delta_{cl}(s)$ = characteristic polynomial of $(A - BK)$ = $|sI - (A - BK)|$

$$\Delta_{cl}(s) = \left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} s - 1 + k_1 & k_2 - 3 \\ -3 & s - 1 \end{pmatrix} \right|$$

$$= s^2 - s + k_1 s - s + 1 - k_1 + 3k_2 - 9 = 0$$

$$= s^2 + (k_1 - 2)s + (3k_2 - k_1 + 8) = 0$$

$k_1 = 3$, $k_2 = -1$

$\Rightarrow$ $s^2 + s + 2 = 0$ $\Rightarrow$ roots = $-0.5 \pm 1.33i$

↳ stable

$k_1 = 4$ $k_2 = 0$

$s^2 + 2s + 4 = 0$ $\Rightarrow$ roots = $-1 \pm \sqrt{3}\, i$

↳ stable

So, which one is better ?

Controllable canonical form

→ Find a transformation matrix $T = P^{-1}$

s.t.

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \qquad \text{for } A \in \mathbb{R}^{4\times 4}$$

$$y = \bar{C}x.$$

has the form

$$\bar{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{bmatrix}$$

$$\bar{g}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

$$= \frac{s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$$T = P^{-1} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$u = -kx$$

How to choose k, such that the closed-loop transfer function has the following form.

$$g_{cl}(s) = \frac{s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}{s^4 + s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}$$

claim:

Choose $\bar{k} = \begin{bmatrix} \eta_1 - \alpha_1, & \eta_2 - \alpha_2, & \eta_3 - \alpha_3, & \eta_4 - \alpha_4 \end{bmatrix}$

and get $k = \bar{k}P$

Proof: check with $\bar{A} - \bar{B}\bar{K}$

$$\bar{B}\bar{K} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \end{bmatrix} = \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A} - \bar{B}\bar{k} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{\bar{k}}(s) = \bar{C}\left(sI - (\bar{A} - \bar{B}\bar{k})\right)^{-1}\bar{B}$$

$$= \frac{s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}{s^4 + s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}$$

Multi-stage optimization problem

$$\min_{x,y,z} \quad f(w, x) + g(x, y) + h(y, z)$$

fix y, and solve innermost problem

$$h^*(y) = \min_z \; h(y, z)$$

then move to next stage.

Fix x and solve for y.

$$g^*(x) = \min_y \; g(x, y) + h^*(y)$$

then move to next stage

Fix w, and solve for x

$$\min_x \quad f(w, x) + g^*(x)$$

→ Connection to LQR

$$\min \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k = \sum_{k=0}^{\infty} l_{QR}(x_k, u_k)$$

$$\min \quad l_{QR}(x_0, u_0) + l_{QR}(x_1, u_1) + l_{QR}(x_2, u_2) + \dots +$$

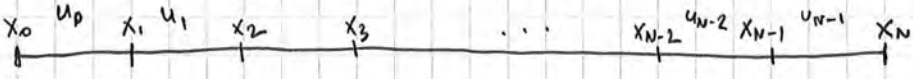
$$x_1 = f(x_0, u_0) = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1$$

Finite horizon:

$$\min \sum_{k=0}^{N-1} l_{QR}(x_k, u_k) + x_N^T P x_N \quad \text{s.t} \quad x_{k+1} = Ax_k + Bu_k$$

Fix x_{N-1}, solve for x_N, u_{N-1} $\quad x_N = Ax_{N-1} + Bu_{N-1}$



DP recursion basic idea.

Sum of quadratic functions

$$V_1(u) = u^T R u \qquad V_2(u) = (Bu - \alpha)^T P (Bu - \alpha)$$

Claim: $V(u) = V_1(u) + V_2(u)$

$$= (u - v)^T H (u - v) + d$$

$$H := R + B^T P B$$

$$v := H^{-1} B^T P \alpha$$

$$d := \alpha^T (P - P^T B H^{-1} B P) \alpha$$

Proof: Expand $V(u)$ and equate to $V_1(u) + V_2(u)$

$$u^T H u - 2u^T H v + v^T H v + d = u^T R u + (Bu - \alpha)^T P (Bu - \alpha)$$

~~$u^T H u - 2u^T H v + v^T H v + d = u^T R u + u^T B^T P B u - 2u^T$~~

$$V_2(u) = (Bu - \alpha)^T P (Bu - \alpha)$$

$$= u^T B^T P B u - 2u^T B^T P \alpha + \alpha^T P \alpha$$

$$u^T H u - 2u^T H v + v^T H v + d = u^T R u + u^T B^T P B u - 2u^T B^T P \alpha + \alpha^T P \alpha$$

$$= u^T (\underbrace{R + B^T P B}_{:= H}) u - 2u^T B^T P \alpha + \alpha^T P \alpha$$

1st term: $u^T H u = u^T (R + B^T P B) u \Rightarrow H = R + B^T P B$

2nd term: $-2u^T H v = -2u^T B^T P \alpha$

$$v = H^{-1} B^T P \alpha$$

3rd term: $v^T H v + d = \alpha^T P \alpha$

$$\alpha^T P^T B H^{-1T} H H^{-1} B^T P \alpha + d = \alpha^T P \alpha$$

$$d = \alpha^T P \alpha - \alpha^T P^T B H^{-1} B^T P \alpha$$

$$= \alpha^T (P - P^T B H^{-1} B^T P) \alpha$$

Let's go back to our finite horizon LQR problem

$$\min \sum_{k=0}^{N-1} \left(\|x_k\|_Q^2 + \|u_k\|_R^2 \right) + \|x_N\|_{P_f}^2$$

$$s.t. \quad x_{k+1} = Ax_k + Bu_k \qquad \forall\, k = \{0, \dots, N-1\}$$

Fix x_{N-1} and solve last stage

$$\min_{x_N,\, u_{N-1}} \|x_{N-1}\|_Q^2 + \|u_{N-1}\|_R^2 + \|x_N\|_{P_f}^2$$

$$s.t. \quad x_N = Ax_{N-1} + Bu_{N-1}$$

$$\min_{x_N,\, u_{N-1}} \|x_{N-1}\|_Q^2 + \underbrace{\|u_{N-1}\|_R^2}_{V_1(u)} + \underbrace{\|Ax_{N-1} + Bu_{N-1}\|_{P_f}^2}_{V_2(u)}$$

$$V_1(u) + V_2(u) = \|u_{N-1} - v\|_H^2 + d$$

~~$H = R + B^T P_f B$~~

$B = B$, $P = P_f$

$\alpha = -Ax_{N-1}$

$$H := R + B^T P_f B$$

$$v := -H^{-1} B^T P_f A x_{N-1}$$

$$d := x_{N-1}^T A^T \left(P_f - P_f^T B H^{-1} B P_f \right) A x_{N-1}$$

$$= x_{N-1}^T \left(A^T P_f A - A^T P_f^T B H^{-1} B P_f A \right) x_{N-1}$$

$$V(x_{N-1}) = \min_{u_{N-1}} \|x_{N-1}\|_Q^2 + \|u_{N-1} - v\|_H^2 + d$$

$u_{N-1} = v$ minimizes this function

$$u_{N-1} = \underbrace{-H^{-1} B^T P_f A}_{:= K} x_{N-1} \qquad u = -Kx$$

$$K = H^{-1} B^T P_f A \qquad H = R + B^T P_f B$$

$$V_{N-1}(x_{N-1}) = \|x_{N-1}\|_Q^2 + d$$

$$= \|x_{N-1}\|_Q^2 + \|x_N\|_{P_f}^2$$

$$V_N(x_{N-1}) = x_{N-1}^T Q x_{N-1} + x_{N-1}^T \left(A^T P_f A - A^T P_f^T B H^{-1} B^T P_f A\right) x_{N-1}$$

$$= x_{N-1}^T \underbrace{\left(Q + A^T P_f A - A^T P_f^T B H^{-1} B^T P_f A\right)}_{:= \Pi_{N-1}} x_{N-1}$$

$$= \|x_{N-1}\|_{\Pi_{N-1}}^2$$

Optimal cost-to-go from x_{N-1} to $x_N = V_{N-1}(x_{N-1}) = \|x_{N-1}\|_{\Pi_{N-1}}^2$

Let's go one step back, with x_{N-2} fixed

$$\min_{x_{N-1}, u_{N-2}} \|x_{N-2}\|_Q^2 + \|u_{N-2}\|_R^2 + V_{N-1}(x_{N-1})$$

$$= \|x_{N-2}\|_Q^2 + \|u_{N-2}\|_R^2 + \|x_{N-1}\|_{\Pi_{N-1}}^2$$

$$= \|x_{N-2}\|_Q^2 + \underbrace{\|u_{N-2}\|_R^2 + \|Ax_{N-2} + Bu_{N-2}\|_{\Pi_{N-1}}^2}$$

$$= \|x_{N-2}\|_Q^2 + \|u_{N-2} - v\|_H^2 + d$$

$$H := R + B^T \Pi_{N-1} B$$

$$v := -H^{-1} B^T \Pi_{N-1} A x_{N-2}$$

$$d := x_{N-2}^T \left(A^T \Pi_{N-1} A - A^T \Pi_{N-1} B H^{-1} B^T \Pi_{N-1} A\right) x_{N-2}$$

$$V_{N-2}(x_{N-2}) = \min_{u_{N-2}} \|x_{N-2}\|_Q^2 + \|u_{N-2} - v\|_H^2 + d$$

minimum occurs when $u_{N-2} = v$

$$u_{N-2} = \underbrace{-H^{-1} B^T \Pi_{N-1} A}_{K} x_{N-2}$$

$$V_{N-2}(x_{N-2}) = \|x_{N-2}\|_Q^2 + d$$

$$= \|x_{N-2}\|_{\Pi_{N-2}}^2, \quad \Pi_{N-2} = Q + A^T \Pi_{N-1} A - A^T \Pi_{N-1} B H^{-1} B^T \Pi_{N-1} A$$

$$\Pi_{k-1} := Q + A^T \Pi_k A - A^T \Pi_k B H^{-1} B^T \Pi_k A$$

$$k = N,\ N-1,\ N-2 \ldots 0$$

with $\Pi_N = P_f$

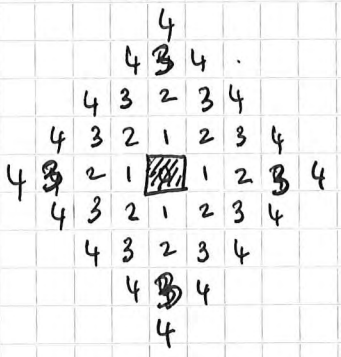
$$u_k = -K_k x_k$$

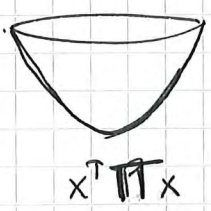
$$K_k = (R + B^T \Pi_{k+1} B)^{-1} B^T \Pi_{k+1} A.$$

For infinite horizon:

$$\Pi = Q + A^T \Pi A - A^T \Pi B H^{-1} B^T \Pi A \rightarrow \text{DARE}$$

$$K = (R + B^T \Pi B)^{-1} B^T \Pi A$$





Reference feedforward

We want $y = r$ as $t \to \infty$

$$u = k_r r - Kx$$

$x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \quad y \in \mathbb{R}^p$

$K \in \mathbb{R}^{m \times n} \quad K_r \in \mathbb{R}^{m \times p}$

$$\dot{x} = Ax + Bu$$
$$= Ax + B(K_r r - Kx)$$
$$= Ax + BK_r r - BKx$$
$$\dot{x} = (A - BK)x + BK_r r$$

At $t \to \infty \quad \dot{x} = 0$ (equilibrium)

$$(A - BK)x_\infty + BK_r r = 0$$
$$x_\infty = -(A - BK)^{-1} BK_r r$$
$$y_\infty = C x_\infty = -C(BK - A)^{-1} BK_r r$$

we want $y_\infty = r$

this means $K_r = [C(BK - A)^{-1}B]^{-1}$

If $C = I$ (i.e., $y = x$)

$u = K(r - x)$ is a common mistake.

Proof: In this case $K = K_r$

But $K = [(BK - A)^{-1} B]^{-1}$

$\Rightarrow \quad BK \neq BK - A$ which cannot be true.

choose $K_r = [(Bk - A)^{-1} B]^{-1}$

i.e., $K_r \neq K$, but depends on K

Introduce deviation variables

$$\bar{x} = x - x_{eq}$$

$$\bar{u} = u - u_{eq}$$

$$\dot{x} = Ax + Bu \quad \text{with} \quad 0 = Ax_{eq} + Bu_{eq}$$

$$\dot{\bar{x}} = A(\bar{x} + x_{eq}) + B(\bar{u} + u_{eq})$$

$$= A\bar{x} + B\bar{u} + \underbrace{Ax_{eq} + Bu_{eq}}_{=0}$$

$$\Rightarrow \quad \dot{\bar{x}} = A\bar{x} + B\bar{u}$$

The system in the deviation variables, has the same dynamics

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{u} = -K\bar{x}$$

$$u = k_r r - kx$$

$$= k_r r - K\bar{x} - Kx_{eq} = \bar{u} + u_{eq}$$

$$\Rightarrow \quad u_{eq} = k_r r - Kx_{eq}$$

Plant: $\dot{x} = Ax + Bu + Bw$

$y = Cx$

integrator state

$$x_a = \int_0^t r(\tau) - Cx(\tau)\, d\tau$$

$\Rightarrow \dot{x}_a = r - Cx$ at equilibrium $\dot{x}_a = 0 \Rightarrow r = Cx$

Augment this with the plant

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

$$u = -\begin{bmatrix} K & -K_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A - BK & BK_a \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

$$\dot{x} = (A - BK)x + BK_a x_a + Bw$$

At steady-state $\dot{x} = 0$, and consider $r = 0$

$$(A - BK)x_\infty + BK_a x_{a,\infty} + Bw_0 = 0$$

$$x_\infty = -(A - BK)^{-1}\left[BK_a x_{a,\infty} + Bw_0\right]$$

$$y_\infty = Cx_\infty = -C(A - BK)^{-1}\left[BK_a x_{a,\infty} + Bw_0\right]$$

$$= -C(A - BK^{-1})B\left[\underline{K_a x_{a,\infty} + w_0}\right] = 0$$

$$K_a x_{a,\infty} = -w_0$$

→ Observability matrix

$x_1 = Ax_0 + Bu_0$, $y_1 = Cx_1$

$y_0 = Cx_0$

$\Rightarrow y_1 = CAx_0 + CBu_0$

$y_2 = CAx_1 + CBu_1$

$= CA(Ax_0 + Bu_0) + CBu_1$

$= CA^2x_0 + CABu_0 + CBu_1$

$y_3 = CAx_2 + CBu_2$

$= CA(A^2x_0 + ABu_0 + Bu_1) + CBu_2$

$= CA^3x_0 + CA^2Bu_0 + CABu_1 + CBu_2$

$y_4 = CAx_3 + CBu_3$

$= CA^4x_0 + CA^3Bu_0 + CA^2Bu_1 + CABu_2 + CBu_3$

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ full rank n

→ Observability Gramian

LTI solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$y(t) = Cx(t) + Du(t)$

$$\Rightarrow y(t) = \underbrace{Ce^{At}x(0)}_{\text{We don't know this natural response}} + \underbrace{C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{We know } u(t), \text{ so we know this}}$$

$$y_{nat}(t) := Ce^{At}x(0)$$

$$e^{A^T t}C^T y_{nat}(t) = e^{A^T t}C^T C e^{At}x(0) \quad \text{(premultiply)}$$

integrate from 0 to t_1

$$\int_0^{t_1} e^{A^T\tau}C^T y_{nat}(\tau)\,d\tau = \underbrace{\int_0^{t_1} e^{A^T\tau}C^T C e^{A\tau}\,d\tau}\; x(0)$$

observability gramian $W_o(t_1)$

invert & premultiply $W_o(t_1)$

$$W_o(t_1)^{-1}\int_0^{t_1} e^{A^T\tau}C^T y_{nat}(\tau)\,d\tau = W_o(t_1)^{-1}W_o(t_1)\,x(0)$$

$$= x(0)$$

$x(0)$ is observable only if $W_o(t_1)$ is invertible

which means $W_o(t_1)$ must have full rank

Output pole vector $y_{p,i}$

→ Compute eigenvalues of A.

→ For each eigenvalue λ_i, get corresponding eigenvector t_i

Eigenvectors should not lie in the nullspace of C

i.e. $y_{p,i} = Ct_i \neq 0 \quad \forall i$

then the system is observable.

$$\underbrace{G(s)}_{\text{proper}} = \underbrace{G_{sp}(s)}_{\text{strictly proper}} + \underbrace{G_\infty}_{\text{constant}}$$

$$G(s) = \overbrace{C(sI - A)^{-1}B}^{\text{strictly proper}} + \overbrace{D}^{\text{constant}}$$

(1) Get $G_\infty = \lim_{s \to \infty} G(s)$

(2) $G_{sp}(s) = G(s) - G_\infty$

$D = G_\infty$

$C(sI - A)^{-1}B = G_{sp}(s)$ $\quad \underline{S^4}$

Controllable canonical form

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad T \rightarrow$$

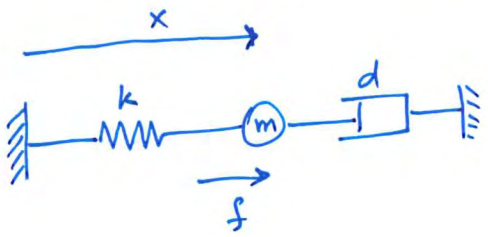
$$C = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} = G_{sp}(s)$$

equate this to $G_{sp}(s)$ to find

n_1, n_2, n_3, n_4 & $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

Example: Mass Spring damper



input $u(t) = f(t)$ force

output $y(t) = x(t)$ position

$$m\ddot{x}(t) + d\dot{x}(t) + k\,x(t) = f(t)$$

Laplace transform

$$m\,X(s)\,s^2 + d\,X(s)\,s + k\,X(s) = f(s)$$

$$X(s)\left[ms^2 + ds + k\right] = f(s)$$

$$G(s) = \frac{X(s)}{f(s)} = \frac{1}{ms^2 + ds + k} = \frac{1/m}{s^2 + \frac{d}{m}s + \frac{k}{m}}$$

$$\lim_{s\to\infty} G(s) = 0$$

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} n_1 & n_2 \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{sn_1 + n_2}{s^2 + s\alpha_1 + \alpha_2} = \frac{0 \cdot s + 1/m}{s^2 + \frac{d}{m}s + \frac{k}{m}}$$

$n_1 = 0 \qquad n_2 = 1/m$

$\alpha_1 = \frac{d}{m} \qquad \alpha_2 = \frac{k}{m}$

$$A = \begin{bmatrix} -d/m & -k/m \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1/m \end{bmatrix} \quad D = 0$$

Example: 2 inputs & 2 outputs

$$G(s) = \begin{bmatrix} \dfrac{-10+4s}{1+2s} & \dfrac{3}{2+s} \\[2ex] \dfrac{1}{(2+s)(1+2s)} & \dfrac{1+s}{(2+s)^2} \end{bmatrix}$$

$$G(s) \Rightarrow \underbrace{G_{sp}(s)}_{C(sI-A)^{-1}B} + \underbrace{G_\infty}_{D}$$

$$\lim_{s\to\infty} G(s) = G_\infty = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_{sp}(s) = G(s) - G_\infty$$

$$= \begin{bmatrix} \dfrac{-10+4s}{1+2s} - 2 & \dfrac{3}{2+s} \\[2ex] \dfrac{1}{(2+s)(1+2s)} & \dfrac{1+s}{(2+s)^2} \end{bmatrix} = \begin{bmatrix} \dfrac{-12+\cancel{4s}-\cancel{4s}}{1+2s} & \dfrac{3}{2+s} \\[2ex] \dfrac{1}{(2+s)(1+2s)} & \dfrac{1+s}{(2+s)^2} \end{bmatrix}$$

Find common denominator

$$d(s) = (1+2s)(2+s)^2$$

$$G_{sp}(s) = \frac{1}{(1+2s)(2+s)^2} \begin{bmatrix} -12(2+s)^2 & 3(1+2s)(2+s) \\ (2+s) & (1+s)(1+2s) \end{bmatrix}$$

$$= \frac{1}{\frac{(1+2s)}{2}(2+s)^2} \begin{bmatrix} \dfrac{-12(s^2+4s+4)}{2} & \dfrac{3(2s^2+5s+2)}{2} \\[2ex] \dfrac{2+s}{2} & \dfrac{2s^2+3s+1}{2} \end{bmatrix}$$

$$= \frac{1}{(s+0.5)(2+s)^2} \begin{bmatrix} -6(s^2+4s+4) & \frac{3}{2}(2s^2+5s+2) \\ \frac{2+s}{2} & \frac{2s^2+3s+1}{2} \end{bmatrix}$$

$$= \frac{1}{(s+0.5)(2+s)^2} \begin{bmatrix} -6s^2-24s-24 & 3s^2+\frac{15}{2}s+3 \\ 1+\frac{s}{2} & s^2+\frac{3}{2}s+\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{(s+0.5)(2+s)^2} \left[\underbrace{\begin{pmatrix} -6 & 3 \\ 0 & 1 \end{pmatrix}}_{=N_1} s^2 + \underbrace{\begin{pmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}}_{=N_2} s + \underbrace{\begin{pmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{pmatrix}}_{=N_3} \right]$$

$$= \frac{1}{s^3+\frac{9}{2}s^2+6s+2} \begin{bmatrix} N_1 s^2 + N_2 s + N_3 \end{bmatrix}$$

Realize this into a controllable canonical form

$$A \equiv \begin{bmatrix} -\alpha_1 I & -\alpha_2 I & -\alpha_3 I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = s^3 + \frac{9}{2}s^2 + 6s + 2$$

$$\alpha_1 = 9/2 \qquad \alpha_2 = 6 \qquad \alpha_3 = 2$$

$$A = \begin{bmatrix} -\frac{9}{2} I & -6I & -2I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

$$C = \left[\begin{pmatrix} -6 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -24 & 15/2 \\ 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} -24 & 3 \\ 1 & 1/2 \end{pmatrix} \right] \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

State Estimation

$\dot{x} = Ax + Bu$

$y = Cx$

Given $y(t)$ & $u(t)$, what is $x(t)$?

x - true state

$\hat{x}$ - estimate

$y_m = y + n$ = measured output with noise n

Estimation error $e := x - \hat{x}$

$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - \dot{\hat{x}}$

we want $e \to 0$, $t \to \infty$

→ if C is square & invertible

$y(t) = Cx(t) \Rightarrow x(t) = C^{-1} y(t)$

→ if we have more measurements than states

i.e. C is a long thin matrix

$\min_x \| y - Cx \|_2^2$ → Least squares estimator

differentiate w.r.t. x and set to 0

$\frac{d}{dx} [(y - Cx)^T (y - Cx)]$

$= \frac{d}{dx} [y^T y - 2 y^T C x + x^T C^T C x]$

$= -2 C^T y + 2 C^T C x = 0$

$x = \underbrace{[C^T C]^{-1} C^T}_{C^\dagger} y$

moore-penrose pseudo inverse of C

Same number or more measurements than states is not realistic in many cases.

Often we have more states than measurements.

Open-loop estimation

$$\dot{x} = Ax + Bu$$

ignore measurements $y(t)$, and simply simulate with $u(t)$

$$\dot{\hat{x}} = A\hat{x} + Bu(t)$$

But - $\hat{x}(0)$ may be unknown / uncertain

- Sensitive to modelling errors
- Slow convergence rate
- May be unstable!

Plant : $\dot{x} = Ax + Bu$

Assume we know A & B exactly.

Open loop estimator : $\dot{\hat{x}} = A\hat{x} + Bu$

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$

$$= Ax + Bu - A\hat{x} - Bu$$

$$= A(x - \hat{x}) = Ae$$

Error dynamics is given by: $\dot{e} = Ae$

Under what conditions is this dynamics stable?

↳ only if A is stable, we have $e \to 0$, $t \to \infty$

⇒ Open-loop estimator cannot be used for unstable plants.

→ Closed-loop estimation

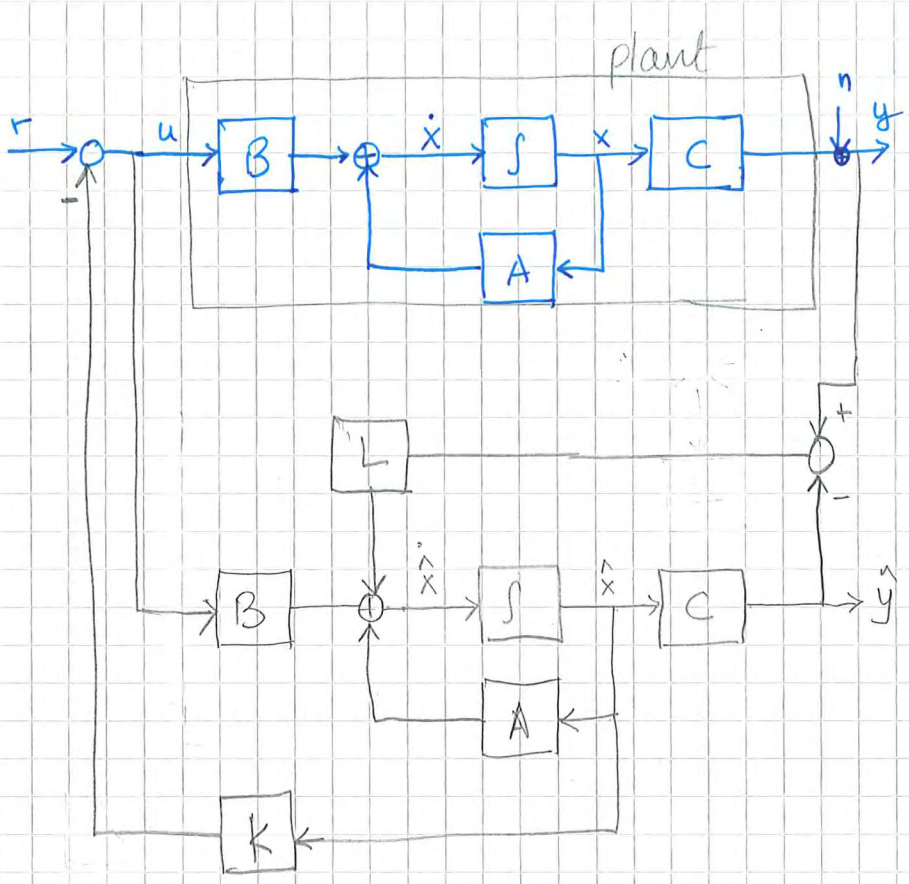
Let us not ignore the measurements $y(t)$

↳ Use open-loop estimator together with feedback from measurement error

measurement error = $y_m - C\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y_m - C\hat{x})$$

⇒ Luenberger Observer



$\dot{x} = Ax + Bu \quad \rightarrow$ plant

$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{L(y_m - \hat{y})}$

$$L(y + n - \hat{y}) = L((Cx + n - C\hat{x})$$
$$= L((Cx - C\hat{x} + n) = LC(x - \hat{x}) + Ln$$
$$= LCe + Ln$$

$\dot{\hat{x}} = A\hat{x} + Bu + LCe + Ln$

Error dynamics

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$
$$= Ax + Bu - A\hat{x} - Bu - LCe - Ln$$
$$= A(x - \hat{x}) + LCe - Ln$$
$$= Ae - LCe - Ln$$
$$= (A - LC)e - Ln$$

eigenvalues of A-LC can be chosen by choosing observer gain matrix L iff (A, C) is observable

Analogous to control design (Controllability - Observability Duality)

if (A, B) is controllable, we can place poles of $(A - BK)$ such that closed loop system is stable.

Similarly, if (A, C) is observable, we can place poles of $(A - LC)$ to make the observer stable

Dual system:

$$\dot{z} = A^T z + C^T u$$

if (A^T, C^T) is controllable, we can choose poles of $(A^T - C^T L^T)$ to make this system stable.

Note: $(A - LC)^T = A^T - C^T L^T$

Therefore we can choose L using the same pole placement method for controller design.

Consider a SISO system

$$\dot{x} = -\frac{1}{T}x + \frac{1}{T}u \qquad y_m = x + n$$

$y = x$ → state is observed, but we want to suppress noise

Closed loop estimator:

$$\dot{\hat{x}} = \underbrace{-\frac{1}{T}\hat{x} + \frac{1}{T}u}_{\text{Copy dynamics}} + \underbrace{L(y_m - C\hat{x})}_{\text{feedback term}}$$

$$\Rightarrow \quad \dot{\hat{x}} = -\frac{1}{T}\hat{x} + \frac{1}{T}u + L(y_m - \hat{x})$$

error dynamics

$$\dot{e} = (A - LC)e - Ln$$

$$\dot{e} = \left(-\frac{1}{T} - L\right)e - Ln$$

We want to place the pole $\lambda_1 = -\frac{1}{T_d}$, choose $L = -\frac{1}{T} + \frac{1}{T_d}$

$$\dot{e} = -\frac{1}{T_d}e + \left(\frac{1}{T} - \frac{1}{T_d}\right)n$$

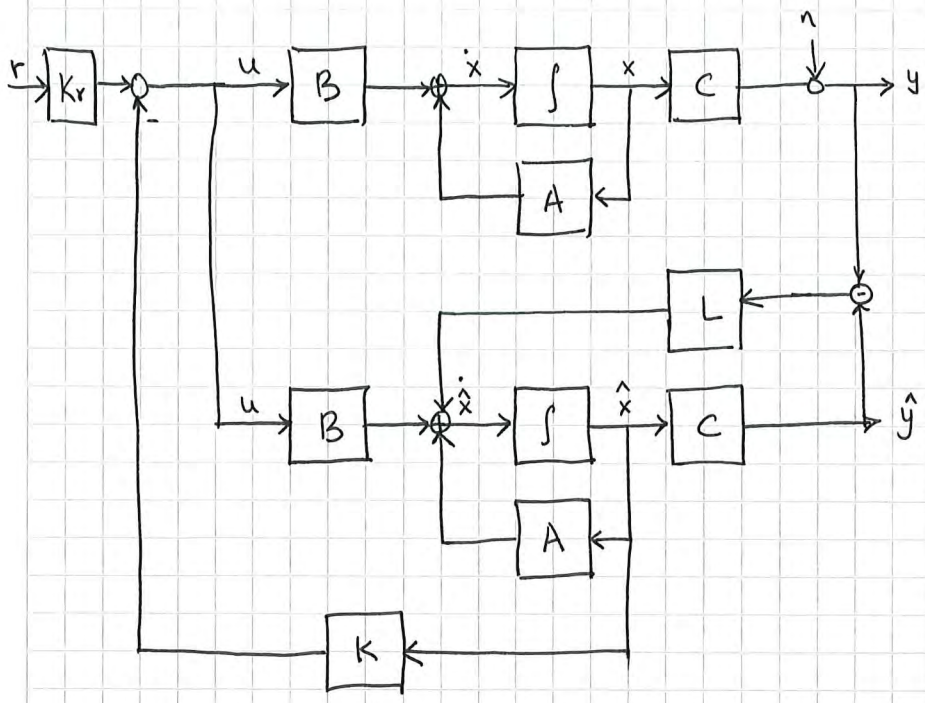
Laplace transform

$$es + \frac{1}{T_d}e = \left(\frac{1}{T} - \frac{1}{T_d}\right)n$$

$$\Rightarrow \quad e\left(s + \frac{1}{T_d}\right) = \left(\frac{1}{T} - \frac{1}{T_d}\right)n$$

$$\Rightarrow \quad \frac{e(s)}{n(s)} = \frac{1/T - 1/T_d}{\left(s + 1/T_d\right)} \Rightarrow \frac{T_d/T - 1}{T_d s + 1}$$

higer T_d → higher cut-off frequency.



Plant: $\dot{x} = Ax + Bu$

Estimator: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

Output feedback: $u = -K\hat{x} + K_r r$, $K_r = [C(BK - A)^{-1}B]^{-1}$

Feedforward gain

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + K_r r) + L(y - C\hat{x})$$

$$= A\hat{x} - BK\hat{x} + BK_r r + Ly - LC\hat{x}$$

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + BK_r r + Ly$$

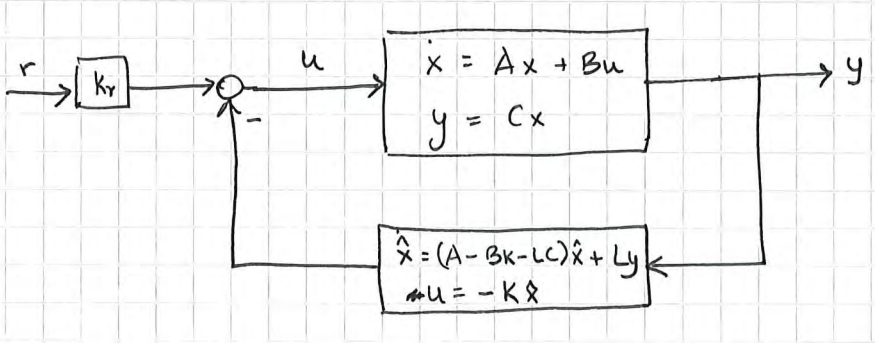
Laplace domain

$$s\hat{x}(s) = (A - BK - LC)\hat{x}(s) + BK_r r(s) + Ly(s)$$

$$\hat{x}(s) = (sI - (A - BK - LC))^{-1}[BK_r r(s) + Ly(s)]$$

$$u(s) = -K\hat{x}(s) + K_r r(s)$$

$$= -K(sI - A + BK + LC)^{-1}[BK_r r(s) + Ly(s)] + K_r r(s)$$



error $e = x - \hat{x}$

Plant: $\dot{x} = Ax + Bu$, $y = Cx$

$u = -K\hat{x} + k_r r = -K(x-e) + k_r r$

$\dot{x} = A\dot{x} + B(-K(x-e)) + k_r r)$

$= Ax + BKx + BKe + Bk_r r$

$= (A-BK)x + BKe + Bk_r r$

$\dot{e} = \dot{x} - \dot{\hat{x}} = (A-LC)e$

closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Laplace transform

$$s\begin{bmatrix} x(s) \\ e(s) \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r(s)$$

$$y(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x(s) \\ e(s) \end{bmatrix} = \begin{bmatrix} sI - A+BK & BK \\ 0 & sI-A+LC \end{bmatrix}^{-1} \begin{bmatrix} Bkr \\ 0 \end{bmatrix} r(s)$$

$$y(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix}$$

$$y(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI - A+BK & BK \\ 0 & sI-A+LC \end{bmatrix}^{-1} \begin{bmatrix} Bkr \\ 0 \end{bmatrix} r(s)$$

$$\Rightarrow \quad y(s) = C(sI - A+BK)^{-1} Bk_r \, r(s)$$

$$\Rightarrow \quad \frac{y(s)}{r(s)} = C(sI - A+BK)^{-1} Bkr \quad = \text{closed-loop transfer function}$$

Stability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

LTI solution

$$x(t) = \underbrace{e^{At}x_0}_{\text{zero input response}} + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{zero state response}} \quad *$$

zero input response → Internal stability

zero state response → input-output stability

Internal stability of a zero input response

$$\dot{x} = Ax \qquad \text{(autonomous system)}$$

Does the system converge to its equilibrium point?

$t \to \infty$, $x \to x_{eq}$ } Asymptotic stability.

Without loss of generality, we can consider $x_{eq} = 0$

$\dot{x} = Ax \longrightarrow$ LTI solution $x(t) = e^{At}x_0$

Eigenvalue condition (sol^n along eigenvectors behave like $e^{\lambda_i t}$)

→ All eigenvalues have negative real parts ⇒ Asymptotic stability

→ All eigenvalues have negative or zero real parts

+ No Jordan block larger than |x1| ⇒ Marginal stability

every finite initial state x_0, produces a bounded response

If these conditions are not met ⇒ Unstable

Exponential stability

$$\exists\ M>0,\ \alpha>0 \quad s.t. \quad \|x(t)\| \leq M e^{-\alpha t}\|x_0\| \quad \forall t \geq 0$$

if an LTI system is asymptotically stable, it is also exponentially stable.

→ each mode $e^{\lambda_i t}$ decays because $Re(\lambda_i) < 0$

→ slowest decaying eigenvalue determines α.

if A has eigenvalues $\lambda_1, \dots, \lambda_n$, the solution along the eigenvectors behaves like $e^{\lambda i t}$

$$0 < \alpha < -\max_i \mathrm{Re}(\lambda_i)$$

Example: $A \in \mathbb{R}^{2\times 2}$

Let λ_1 and λ_2 be its eigenvalues, and q_1, q_2 be its corresponding eigenvectors

General solution of $\dot{x} = Ax$ is a linear combination of eigenvectors

$$x(t) = c_1 e^{\lambda_1 t} q_1 + c_2 e^{\lambda_2 t} q_2$$

where c_1 & c_2 are determined by the initial state x_0

at $t=0$, $x_0 = c_1 q_1 + c_2 q_2 = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Q \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Q^{-1} x_0$$

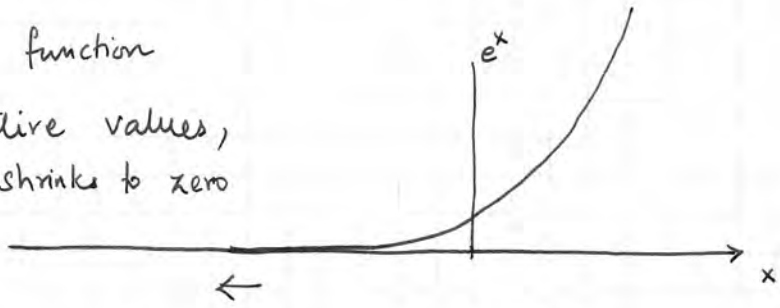
Solution $x(t)$ along $q_1 = c_1 e^{\lambda_1 t} q_1$ #

" " " $q_2 = c_2 e^{\lambda_2 t} q_2$

Solution $x(t)$ is a linear combination $\Rightarrow x(t) = c_1 e^{\lambda_1 t} q_1 + c_2 e^{\lambda_2 t} q_2$

Exponential function

For negative values, this shrinks to zero



$\Rightarrow c_1 e^{\lambda_1 t} q_1$ shrinks to zero if $\lambda_1 < 0$

For a general $A \in \mathbb{R}^{n\times n}$

$$x(t) = c_1 e^{\lambda_1 t} q_1 + c_2 e^{\lambda_2 t} q_2 + \dots + c_n e^{\lambda_n t} q_n$$

$$= \sum_{i=1}^{n} c_i e^{\lambda i t} q_i$$

→ each term → 0, if $\mathrm{Re}(\lambda_i) < 0$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Q^{-1} x_0$$

→ $\mathrm{Re}(\lambda_i) < 0 \Rightarrow$ decay along q_i

→ $\mathrm{Re}(\lambda_i) > 0 \Rightarrow$ grow along q_i

→ $\mathrm{Im}(\lambda_i) \neq 0 \Rightarrow$ spiral along q_i

Equivalence transformation

$$\dot{x} = Ax + Bu$$

↓ Diagonalization

find eigenvalues of A, and corresponding eigenvectors

$$Q = [q_1 \; q_2 \cdots q_n]$$

Equivalence transform $x = Q\bar{x}$ (i.e., $T = Q^{-1}$)

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \qquad \bar{A} = Q^{-1}AQ \;, \quad \bar{B} = Q^{-1}B$$

Consider zero input response, i.e; $u = 0$

$$\dot{\bar{x}} = \bar{A}\bar{x} \quad \rightarrow \text{LTI solution}$$

$$\bar{x}(t) = e^{\bar{A}t}\bar{x}_0 \qquad \bar{A} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix}$$

Functions of square matrix

if $\bar{A}$ is diagonal, $f(A) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$

$$\hookrightarrow e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \; e^{\lambda_n t} \end{bmatrix}$$

$$\bar{x}(t) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \; e^{\lambda_n t} \end{bmatrix} \bar{x}_0$$

pre-multiply with the eigenvectors $Q = [q_1, q_2 \cdots q_n]$

$$Q\bar{x}(t) = Q \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \; e^{\lambda_n t} \end{bmatrix} \bar{x}_0$$

$$= [q_1 \; q_2 \cdots q_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \; e^{\lambda_n t} \end{bmatrix} \bar{x}_0$$

$$\Rightarrow x(t) = \begin{bmatrix} q_1 e^{\lambda_1 t} & q_2 e^{\lambda_2 t} & \cdots & q_n e^{\lambda_n t} \end{bmatrix} \bar{x}_0$$

Noting that $\bar{x}_0 = Q^{-1}x_0$, we define this as $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$\Rightarrow x(t) = q_1 e^{\lambda_1 t} c_1 + q_2 e^{\lambda_2 t} c_2 + \cdots + q_n e^{\lambda_n t} c_n$$

Repeated eigenvalues → Jordan form

$$A = P J P^{-1} \quad \text{with} \quad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$x(t) = e^{At} x_0$$

$$e^{At} = P e^{Jt} P^{-1} \qquad (\text{because } f(PAP^{-1}) = P f(A) P^{-1})$$

$$e^{Jt} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

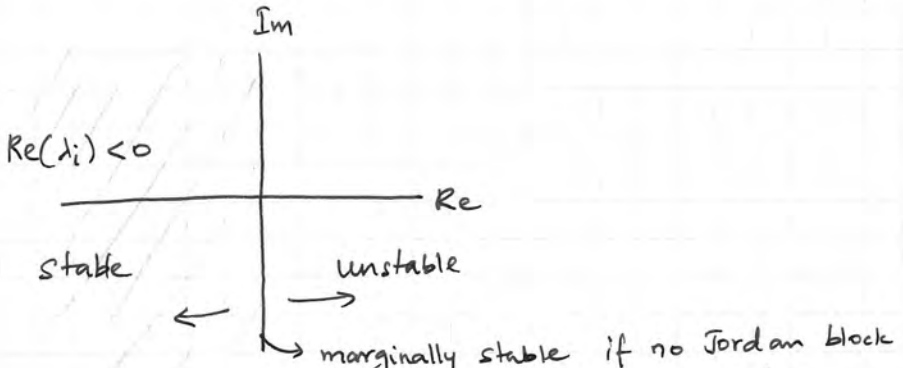
if $\mathrm{Re}(\lambda) < 0 \Rightarrow x(t) = 0$ as $t \to \infty$ $\Rightarrow$ stability

if $\mathrm{Re}(\lambda) = 0$

$$x(t) = C_1 e^0 + C_2 t e^0 \Rightarrow C_1 + C_2 t$$

$\Rightarrow x(t) \to \infty$, as $t \to \infty$ $\Rightarrow$ unstable

$\mathrm{Im}(\lambda_i) \neq 0$
↳ spiral along q_i



Discrete time system

$$x_{k+1} = A x_k$$

Assume $A \in \mathbb{R}^{n \times n}$ is diagonalizable with n linearly independent eigenvectors $q_1, \dots q_n$ and $\lambda_1 \dots \lambda_n$ be their corresponding eigenvalues

Solution along eigenvectors

$$x_1 = A x_0 = c_1 \lambda_1 q_1 + c_2 \lambda_2 q_2 + \dots + c_n \lambda_n q_n$$

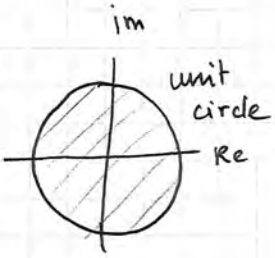
$$x_2 = A x_1 = A^2 x_0 = c_1 \lambda_1^2 q_1 + c_2 \lambda_2^2 q_2 + \dots + c_n \lambda_n^2 q_n$$

By induction

$$x_k = c_1 \lambda_1^k q_1 + c_2 \lambda_2^k q_2 + \dots + c_n \lambda_n^k q_n = \sum_{i=1}^{n} c_i \lambda_i^k q_i$$

if $|\lambda_i| < 1 \quad \forall i \Rightarrow$ stability

$|\lambda_i| > 1 \Rightarrow$ growth along $q_i \Rightarrow$ unstable



Complex $\lambda_i \Rightarrow$ Spiral behaviour.

$|\lambda_i| = 1 \Rightarrow$ marginally stable

Suspended Magnet example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.04 \\ 0 \end{bmatrix} u$$

internal stability $\Rightarrow$ zero input response, i.e., $u = 0$

eigenvalues of $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ $\quad |\lambda I - A| = 0$

$$\lambda_1 = 2, \quad \lambda_2 = -2$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{vmatrix} \lambda & -4 \\ -1 & \lambda \end{vmatrix} \Rightarrow \quad \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

$$A q_i - \lambda_i q_i = 0 \quad \Rightarrow (A - \lambda_i I) q_i = 0$$

$\lambda_1 = 2$

$$\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) q_1 = 0$$

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} q_1 = 0 \Rightarrow \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0$$

$$\Rightarrow \quad -2 q_{11} + 4 q_{12} = 0 \quad \Rightarrow q_1 \propto \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

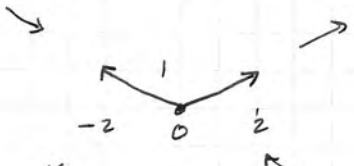
$$q_{11} - 2 q_{12} = 0$$

$\lambda_2 = -2 \qquad (A - \lambda_i I)\, q_i = 0$

$$\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right) q_{t2} = 0$$

$$= \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} q_{t2} = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = 0$$

$$\Rightarrow \quad q_{t2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Solution along $q_{t1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow C_1 e^{\lambda_1 t} q_1$

$\lambda_1 = 2 > 0 \Rightarrow$ Solution grows as $t \to \infty$

Solution along $q_{t2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow C_2 e^{\lambda_2 t} q_2$

$\lambda_2 = -2 < 0 \rightarrow$ Solution shrinks to zero as $t \to \infty$

$X(t) =$ Linear Combination of $C_1 e^{\lambda_1 t} q_1 + C_2 e^{\lambda_2 t} q_2$

Lyapunov stability

A symmetric matrix $P = P^T$ is positive definite, i.e. $P \succ 0$

if $x^T P x > 0 \quad \forall x \neq 0$

& positive semi-definite, $P \succcurlyeq 0$

if $x^T P x \geq 0$

↳ All eigenvalues of P are positive

↳ leading principle minors are all positive

if $P \succ 0$, $\exists$ a nonsingular $n \times n$ matrix L, such that $P = L^T L$

Lyapunov stability is a formal method to verify stability without explicitly solving the system.

imagine the Lyapunov function $V(x)$ represents some form of energy

- Energy is lowest at the equilibrium point
- Energy decreases along all paths (dissipative)
  ⇒ trajectories converge to the equilibrium

A Lyapunov function $V(x)$ is any positive $\overset{\text{def.}}{\wedge}$ function

- $V(0) = 0$ & $V(x) > 0 \quad \forall x \neq 0$ (positive definite)
- $\dot{V}(x) \leq 0$ (non-decreasing along all trajectories)

Simplest Lyapunov function is a quadratic function

$V(x) = \frac{1}{2} x^T P x$, where $P \succ 0$

↳ this satisfies the first condition

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = x^T P A x$$

$$= \frac{1}{2} x^T A^T P x + \frac{1}{2} x^T P A x$$

$$= \frac{1}{2} x^T (A^T P + P A) x$$

define $-N := A^T P + P A$

$$\dot{V}(x) = -\frac{1}{2} x^T N x$$

if $N \succ 0$, then $\dot{V}(x) < 0$ ⇒ ~~but~~ energy dissipative ⇒ system is stable

Lyapunov equation

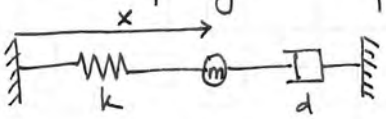
$$A^T P + PA = -N$$

Lyapunov stability condition

→ For a given $N \succ 0$, if the Lyapunov equation has a unique solution $M \succ 0$, then the system is asymptotically stable

→ Note: the reverse test is not reliable.

Example: Mass-Springer damper



$$X = \begin{bmatrix} \text{position } x \\ \text{Velocity } \dot{x} \end{bmatrix}$$

$$\dot{X} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix}}_{A} X$$

$$E = KE + PE = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} X^T \underbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}_{:= P} X$$

$$V(x) = \text{Energy} = \frac{1}{2} x^T P x$$

$$\dot{V}(x) = \dot{E} = \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x}$$

$$= \frac{1}{2} x^T A^T P x + \frac{1}{2} x^T P A x = \frac{1}{2} x^T (A^T P + PA) x$$

$$(A^T P + PA) = \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix} = -N$$

$$\Rightarrow \dot{V}(x) = \dot{E} = -\frac{1}{2} x^T N x$$

$N \succcurlyeq 0$ (but not positive definite, since one of the eigenvalues = 0)

⇒ system is marginally stable (?)

But we know that the system is stable!

Checking for positive definiteness of N, given some M is not reliable.

Choose $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Can we find $P > 0$, such that

$$A^T P + PA = -N$$

Let $k=1$, $d=1$, $m=1$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$ note P must be symmetric.

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -p_2 & -p_3 \\ p_1 - p_2 & p_2 - p_3 \end{bmatrix} + \begin{bmatrix} -p_2 & p_1 - p_2 \\ -p_3 & p_2 - p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2p_2 & p_1 - p_2 - p_3 \\ p_1 - p_2 - p_3 & 2p_2 - 2p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad -2p_2 = -1 \Rightarrow p_2 = \frac{1}{2}$$

$$2p_2 - 2p_3 = -1 \Rightarrow 1 - 2p_3 = -1 \Rightarrow p_3 = 1$$

$$p_1 - p_2 - p_3 = 0 \qquad p_1 = p_2 + p_3 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\Rightarrow P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Check if M is positive definite

Leading principle minors $= \{1.5, |P|\}$

$$= \{1.5, \ 1.5 - 0.25\} > 0$$

$\Rightarrow P > 0$, hence system is stable in the sense of Lyapunov.

Matlab: $P = \text{lyap}(A, N)$ solves $A^T P + PA = -N$

in discrete time

$P = \text{dlyap}(A, N)$ solves $A^T P A - P = -N$

Lyapunov function in Discrete time

$$x_{k+1} = A x_k$$

$$V(x_k) = x_k^T P x_k \quad , \quad P \succ 0$$

$$\Delta V = V(x_{k+1}) - V(x_k) < 0 \quad \text{for stability}$$

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k$$

$$x_k^T A^T P A x_k - x_k^T P x_k$$

$$\Rightarrow x_k^T (A^T P A - P) x_k < 0$$

Lyapunov condition

$$A^T P A - P = -N \quad \text{for some } N \succ 0$$

Stability of LQR

$$V(x_k) : x_k^T P x_k$$

Here P is the solution of the discrete Algebraic Riccati eqn (DARE)

$$P = A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A + Q$$

For stability we need

$$\Delta V = V(x_{k+1}) - V(x_k) \quad , \text{ where } x_{k+1} : (A - BK) x_k$$

$$\Delta V = x_k^T (A - BK)^T P (A - BK) x_k - x_k^T P x_k$$

$$= x_k^T \left((A^T - K^T B^T) P (A - BK) - P \right) x_k$$

$$= x_k^T \left(A^T P A - A^T P B K - K^T B^T P A + K^T B^T P B K - P \right) x_k$$

Substitute K from the Riccati solution

$$K = (R + B^T P B)^{-1} B^T P A$$

This gives

$$\Delta V = x_k^T \underbrace{\left(-(Q + K^T R K) \right)}_{:= N} x_k$$

Define $N = Q + K^T R K \succ 0$, then the closed-loop system

$$\Delta V = -x_k^T N x_k < 0 \quad ,$$

thence the LQR controller is a stabilizing control law.

Input-Output Stability (A,B) controllable & (A,C) observable

Recall, the LTI solution

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{zero input response}} + \underbrace{C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t)}_{\text{zero state response.}}$$

Bounded input : An input $u(t)$ is bounded if $\exists$ a constant u_m such that $|u(t)| \leq u_m < \infty \quad \forall\, t \geq 0$

How does the output respond to a bounded input

i.e. is the System BIBO stable?

if $|u(t)| \leq u_m < \infty$, $\forall t \Rightarrow |y(t)| \leq M < \infty$?

Suppose $|u(t)| \leq u_m < \infty$, $\forall t$, then

$$|y(t)| = \left| \int_0^t Ce^{A(t-\tau)}B\,u(\tau)\,d\tau \right|$$

Using the triangle inequality

$$|y(t)| \leq \int_0^t |Ce^{A(t-\tau)}B|\,|u(\tau)|\,d\tau$$

$$\leq u_m \int_0^t |Ce^{A(t-\tau)}B|\,d\tau$$

~~if Any integral $\int_0^\infty |e^{A(t-\tau)}B|$~~

if the integral $\int_0^\infty |Ce^{At}B|\,dt \leq M < \infty$, then we have BIBO stability

→ $\int_0^\infty |Ce^{At}B|$ measures the total gain the system can apply to the input over time.

↳ if the system is stable, i.e. all eigenvalues of A have negative real parts → e^{At} decays → integral is finite

⇒ Bounded output

↳ if A has an eigenvalue with positive real part → e^{At} grows

⇒ integral diverges ⇒ BIBO unstable

In short, decay of e^{At} ensures finite integral and hence BIBO stability.

BIBO stability in discrete time

$x_0 = 0$ initial state = 0

$x_1 = Ax_0 + Bu_0 = Bu_0$

$x_2 = Ax_1 + Bu_2 = ABu_0 + Bu_2$

$x_3 = Ax_2 + Bu_2 = A^2Bu_0 + ABu_1 + Bu_2$

$\vdots$

$$x_k = Ax_{k-1} + Bu_{k-1} = \sum_{i=0}^{k-1} A^{k-1-i} B u_i$$

$$y_k = Cx_k = \sum_{i=0}^{k-1} CA^{k-1-i} B u_i$$

$$|y_k| = \left| \sum_{i=0}^{k-1} CA^{k-1-i} B u_i \right| \leq \sum_{i=0}^{k-1} |CA^{k-1-i}B||u_i|$$

if the input is bounded, i.e. $|u_k| \leq u_m < \infty$

$$|y_k| \leq u_m \sum_{i=0}^{k-1} |CA^{k-1-i}B|$$

if the sum of absolute values of the impulse response is finite, i.e $\sum_{i=1}^{\infty} |CA^{k-1-i}B| \leq M < \infty$, then we have BIBO stability.

→ CA^iB measures the contribution of past inputs $u_0, u_1 \ldots$ to the output at step k.

if A is stable, i.e $|\lambda_i| < 1$, $\forall i$, this implies that

$A^i \to 0$ → finite sum ⇒ bounded output

if A has some eigenvalue $|\lambda_i| \geq 1$, A^i grows, → infinite sum

⇒ BIBO unstable.

There is no such thing as BIBO marginal stability.

→ only BIBO stable or BIBO unstable.

in Continous time: for BIBO stability we need

$$\int_0^\infty |C e^{At} B| \, dt \leq M < \infty$$

if A has an eigenvalue with zero real part, e^{At} does not decay.

$\Rightarrow \int_0^\infty$ finite number $= \infty \Rightarrow$ BIBO unstable

in discrete time: we need

$$\sum_{i=0}^{\infty} |C A^{k-1-i} B| \leq M < \infty$$

if $|\lambda_i| = 1$, A^i does not decay,

$\sum_{i=0}^{\infty}$ finite number $= \infty \Rightarrow$ BIBO unstable

	Continous time	discrete time	internal stability		BIBO
→	$Re(\lambda_i) < 0, \forall i$	$\lvert\lambda_i\rvert < 1 \ \forall i$	asymptotic		✓
→	$Re(\lambda_i) = 0$	$\lvert\lambda_i\rvert = 1$	$\Rightarrow$ marginal	$\Rightarrow$	X
→	$Re(\lambda_i) < 0$	$\lvert\lambda_i\rvert > 1$	unstable		X

Recap: stability

	Continuous time	discrete Time	Internal stability		BIBO
→	$Re(\lambda_i) < 0, \forall i$	$\lvert\lambda_i\rvert < 1, \forall i$	asymptotic		✓
→	$Re(\lambda_i) = 0$	$\lvert\lambda_i\rvert = 1$	marginal	$\Rightarrow$	×
→	$Re(\lambda_i) > 0$	$\lvert\lambda_i\rvert > 1$	unstable		×

→ eig(A) are important

internal stability $\Rightarrow$ BIBO stability

Note! But BIBO stability $\not\Rightarrow$ internal stability.

The answer lies in observability & controllability.

internal stability deals with $\dot{x} = Ax$.

input-output stability deals with $y = \left(C(sI - A)^{-1}B + D\right)u$

$$y(t) = C\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau + D\,u(t)$$

iff (A, B) is controllable & (A, C) is observable,

then BIBO stability $\Rightarrow$ internal stability.

Why? Because only if (A, B) is controllable, (A, C) is observable all the modes (given by eigenvalues λ_i) would contribute to the input-output map $C(sI - A)^{-1}B$

and this means, there are no hidden unstable modes.

Uncontrollable & unobservable states cannot be stabilized!

→ If (A, B) is not controllable:

↳ then there exists a subspace $x_{\bar{c}}$ with

$$\dot{x}_{\bar{c}} = A_{\bar{c}} x_{\bar{c}}, \quad B_{\bar{c}} = 0$$

i.e, input u cannot affect $x_{\bar{c}}$

→ If (A, C) is not observable:

↳ then there exists a subspace $x_{\bar{o}}$ with

$$y = C_{\bar{o}} x_{\bar{o}} \quad \text{where} \quad C_{\bar{o}} = 0$$

i.e. y is not affected by $x_{\bar{o}}$

This idea of subspaces $X_{\bar{c}}$ & $X_{\bar{o}}$ gives rise to the concept of Kalman decomposition

Example: 2x2 case

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix}}_{\bar{B}} u \quad , \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{12} & \bar{c}_{2n} \end{bmatrix}}_{=\bar{C}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Controllability matrix $C = [\bar{B} \quad \bar{A}\bar{B}]$

$$C = \left[\underbrace{\begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix}}_{\bar{B}} \quad \underbrace{\begin{pmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{pmatrix} \begin{pmatrix} \bar{b}_c \\ 0 \end{pmatrix}}_{\bar{A}\bar{B}} \right]$$

$$= \begin{bmatrix} \bar{b}_c & \bar{a}_c \bar{b}_c \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{b}_c & \bar{a}_c \bar{b}_c \\ 0 & 0 \end{bmatrix}$$

$\bar{x}_2$ is uncontrollable

Transfer function $G(s) = \bar{C}(s\bar{I} - \bar{A})^{-1}\bar{B}$

$$G(s) = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{12} & \bar{c}_{2n} \end{bmatrix} \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{pmatrix} \right]^{-1} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{12} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \frac{1}{s-\bar{a}_c} & \text{(squiggle)} \\ 0 & \frac{1}{s-\bar{a}_{nc}} \end{bmatrix} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{12} & \bar{c}_{21} \end{bmatrix} \begin{bmatrix} \frac{\bar{b}_c}{s-\bar{a}_c} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{\bar{c}_{11}\bar{b}_c}{s-\bar{a}_c} \\ \frac{\bar{c}_{12}\bar{b}_c}{s-\bar{a}_c} \end{bmatrix} u$$

$$y_1 = \frac{\bar{c}_{11}\bar{b}_c}{s-\bar{a}_c} u \qquad y_2 = \frac{\bar{c}_{12}\bar{b}_c}{s-\bar{a}_c} u$$

All information about uncontrollable state is gone.

Example 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Controllability matrix = $[B \quad AB]$

$$\Rightarrow AB = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}$$ which is rank deficient

$$\dot{x}_1 = 2x_1 + x_2 + 3u$$

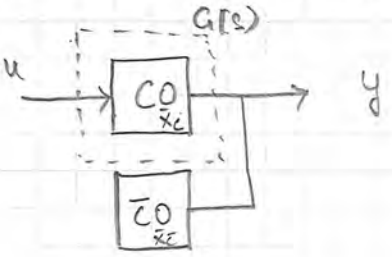
$$\dot{x}_2 = -x_2$$

↑ Neither x_1 nor u influences x_2.

A more general case: decompose into controllable & uncontrollable states

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$



Controllability matrix $C = \begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

with rank $\rho(C) = n_1 < n$

Transfer function matrix $G(s) = \bar{C}_c(s\bar{I} - \bar{A}_c)^{-1}\bar{B}_c + D$

zero state equivalence:

$$\dot{\bar{x}}_c = \bar{A}_c\bar{x}_c + \bar{B}_c u$$
$$y_s = \bar{C}_c\bar{x}_c + Du$$

$\{A, B, C, D\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are zero state equivalent if they have the same transfer function.

NB! Realization of transfer function can only produce the controllable (and observable) subsystem.

Consider another example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is this system controllable?

$$C = [B \;\; AB] = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

which is rank deficient and hence not controllable!

How do we decompose this into $\bar{x}_c$ and $\bar{x}_{\bar{c}}$?

→ Equivalence tranform

Find a transformation matrix $T = Q^{-1}$

such that $x = Q\bar{x}$, $\bar{A} = Q^{-1}AQ$, $\bar{B} = Q^{-1}B$, $\bar{C} = CQ$

How to compose Q matrix?

→ use the linearly independent columns of the controllability matrix.

if $\text{rank}(C) = n_1 < n$,

$$Q = [\underbrace{q_1 \; q_2 \; \cdots \; q_{n_1}}_{\text{linearly independent columns of } C \text{ i.e. span}(C)} \; \underbrace{\cdots \; q_n}_{\text{chosen arbitrarily to ensure } Q \text{ is invertible}}]$$

Back to our example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$C = [B \quad AB] = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

↑ 2nd column = 2 × 1st column

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

↳ chosen arbitrarily s.t. Q is invertible

one choice = null(C^T)

$\bar{x}_c = x_1 + x_2$ ← $\bar{x}_c$ controllable state

$\bar{x}_{\bar{c}} = x_1 - x_2$ ← $\bar{x}_{\bar{c}}$ uncontrollable state

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix}$$

$$\left.\begin{aligned} \dot{\bar{x}}_{\bar{c}} &= \bar{A}_{\bar{c}}\,\bar{x}_{\bar{c}} + \bar{B}_{\bar{c}}\,u \\ y &= \bar{C}_{\bar{c}}\,\bar{x}_{\bar{c}} \end{aligned}\right\} \quad \bar{A}_{\bar{c}} = 2 \qquad \bar{B}_{\bar{c}} = 0 \qquad \bar{C}_{\bar{c}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Decompose the states into observable & unobservable states

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\end{bmatrix} = \begin{bmatrix}\bar{a}_0 & 0\\ \bar{a}_{21} & \bar{a}_{no}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} + \begin{bmatrix}\bar{b}_1\\ \bar{b}_{no}\end{bmatrix} u$$

$$y_1 = \begin{bmatrix}\bar{C}_1 & 0\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}$$

observability matrix $\mathcal{O} = \begin{bmatrix}C\\ CA\end{bmatrix}$

$$= \begin{bmatrix}\bar{C}_1 & 0\\ \bar{C}_1\bar{a}_0 & 0\end{bmatrix}$$ which is rank deficient $\Rightarrow$ $\bar{x}_2$ is unobservable.

Transfer function

$$y_1(s) = \begin{bmatrix}\bar{C}_1 & 0\end{bmatrix}\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix} - \begin{pmatrix}\bar{a}_0 & 0\\ \bar{a}_{21} & \bar{a}_{no}\end{pmatrix}\right]^{-1}\begin{bmatrix}\bar{b}_1\\ \bar{b}_{no}\end{bmatrix} u(s)$$

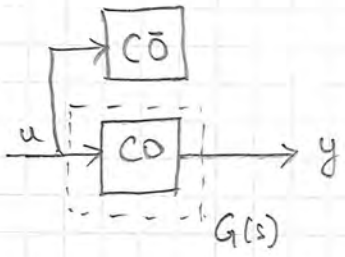
$$y(s) = \begin{bmatrix}\bar{C}_1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{s-\bar{a}_0} & 0\\ \sim & \frac{1}{s-\bar{a}_{no}}\end{bmatrix}\begin{bmatrix}\bar{b}_1\\ \bar{b}_{no}\end{bmatrix} u(s)$$

$$= \begin{bmatrix}\frac{\bar{C}_1}{s-\bar{a}_0} & 0\end{bmatrix}\begin{bmatrix}\bar{b}_1\\ \bar{b}_{no}\end{bmatrix} u(s) \Rightarrow y(s) = \frac{\bar{C}_1\bar{b}_1}{s-\bar{a}_0} u(s)$$

General case

$$\begin{bmatrix}\dot{\bar{x}}_o\\ \dot{\bar{x}}_{\bar{o}}\end{bmatrix} = \begin{bmatrix}\bar{A}_o & 0\\ \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}}\end{bmatrix}\begin{bmatrix}\bar{x}_o\\ \bar{x}_{\bar{o}}\end{bmatrix} + \begin{bmatrix}\bar{B}_o\\ \bar{B}_{\bar{o}}\end{bmatrix} u$$

$$y = \begin{bmatrix}\bar{C}_o & 0\end{bmatrix}\begin{bmatrix}\bar{x}_o\\ \bar{x}_{\bar{o}}\end{bmatrix} + Du$$



Observability matrix $\mathcal{O} = \begin{bmatrix}\bar{C}_o & 0\\ \bar{C}_o\bar{A}_o & 0\end{bmatrix}$

with $\text{rank}(\mathcal{O}) = n_2 < n$

Transfer function: $G(s) = \bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o + D$

Equivalence transformation to ~~transform~~ decompose the system into observable & unobservable states

Transformation matrix $T = P$ is constructed using linearly independent rows of $\mathcal{O}$

$$\bar{x} = Px \quad , \quad \bar{A} = PAP^{-1} \quad , \quad \bar{B} = PB \quad , \quad \bar{C} = CP^{-1}$$

where $P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n_2} \\ \vdots \\ p_n \end{bmatrix}$ $\left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\}$ linearly independent rows of $\mathcal{O}$ i.e. span($\bar{\mathcal{O}}$)

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ arbitrarily chosen s.t. P is invertible e.g. null ($\mathcal{O}$)

General case combining the controllability and observability to decompose the system into four subsystems:

→ x_{co} - states that are controllable and observable

→ $x_{c\bar{o}}$ → states that are controllable, but not observable

→ $x_{\bar{c}o}$ → states that are observable, but not controllable

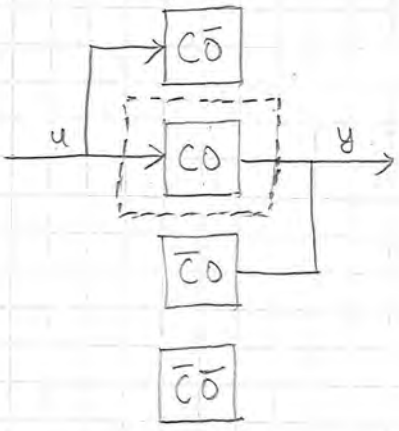
→ $x_{\bar{c}\bar{o}}$ → states that are neither controllable, nor observable.

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & & \bar{A}_{13} & \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ & & \bar{A}_{\bar{c}o} & \\ & & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

Transfer matrix $G(s) = \bar{C}_{co}(s\bar{I} - \bar{A}_{co})^{-1} B_{co} + D$

Kalman Decomposition



→ state-space is an internal description of the system
↳ relationship between input, output & its internal variables

→ Transfer function is an external description of the system
↳ has no information about the inside of the system.

Any state space can be converted into the Kalman decomposition by choosing a suitable transformation matrix based on the observability & controllability matrices

$R_c = \text{span}(C)$ $\quad R_o = \text{span}(\mathcal{O}^T)$

$R_{\bar{c}} = \text{null}(C^T)$ $\quad R_{\bar{o}} = \text{null}(\mathcal{O})$

$R_{co} = R_c \cap R_o$

$R_{c\bar{o}} = R_c \cap R_{\bar{o}}$

$R_{\bar{c}o} = R_{\bar{c}} \cap R_o$

$R_{\bar{c}\bar{o}} = R_{\bar{c}} \cap R_{\bar{o}}$

$Q^{-1} = T = [R_{co} \quad R_{c\bar{o}} \quad R_{\bar{c}o} \quad R_{\bar{c}\bar{o}}]$

Kalman decomposition is extremely useful for analysis & design
Focus on Co modes for closed-loop stability.

→ Minimal realization keeps only the Co states

Minimal realization is given by

$$\dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u$$

$$y = \bar{C}_{co}\bar{x}_{co} + Du$$

→ A state-space system (A, B, C, D) is a minimal realization iff (A, B) is controllable and (A, C) is observable

→ if a system (A, B, C, D) is a minimal realization
then asymptotic internal stability $\Leftrightarrow$ BIBO stability

because there are no hidden modes in a minimal realization

State-space is an internal description, and transfer function is an external description.

→ Q: Does the transfer function describe fully a system?

Given a system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$
$y = Cx$

The system is completely characterized by its transfer function, iff:
- the transfer function is co-prime (i.e. no common roots between $N(s)$ & $D(s)$)
- $\deg(D(s))$ = no. of states n = $\dim(A)$

Note: for co-prime fractions $g(s) = \frac{N(s)}{D(s)}$

$$\deg(g(s)) = \deg(D(s))$$

To summarize:

→ A state-space system (A, B, C, D) is a minimal realization of a proper rational transfer function $g(s)$ iff

(A, B) is controllable & (A, C) is observable

or

iff $\dim A = \deg(g(s))$

and $g(s)$ completely characterizes the system (A, B, C, D)

For these systems

asymptotic internal stability $\Leftrightarrow$ input-output stability

Example: Is the following system BIBO stable & internally stable?

$$\dot{x}(t) = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 7 & 8 \end{bmatrix} x + 1.5\,u$$

Internal stability:

check eig(A)

Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = 3$

$\Rightarrow$ System is not internally stable.

BIBO stability

first check whether (A, B) & (A, C) are controllable & observable

Controllability matrix

$$C = [B \quad AB] = \begin{bmatrix} 4 & -8 \\ 0 & 0 \end{bmatrix}$$

$\text{rank}(C) = 1 < 2$

$\Rightarrow$ uncontrollable

observability matrix

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -14 & (35+28) \end{bmatrix}$$

$\text{rank}(O) = 2$

$\Rightarrow$ observable.

Controllable subsystem:

$$\dot{x}_1 = -2x_1 + 4u$$

$$y = 7x_1 + 1.5u$$

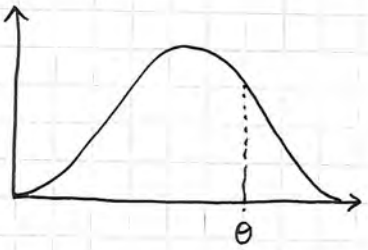
i.e. $A_c = -2$

$$\text{eig}(A_c) = -2 \Rightarrow \text{BIBO stable.}$$

$\Rightarrow$ Therefore the system is input-output stable, but not internally stable.

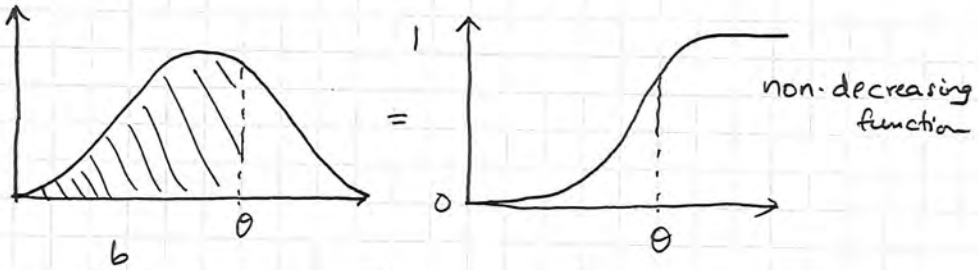
Random variable : single uncertain quantity

Probability density function (PDF) = $f_x(\theta)$ → shape of the curve

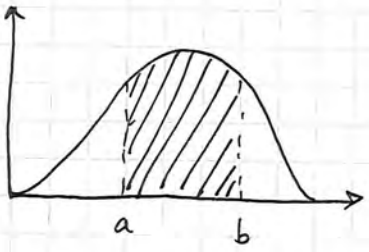


Cumulative density function (CDF) = $F_x(\theta) = \mathbb{P}(X \leq \theta) = \int_{-\infty}^{\infty} f_x(\theta)\, d\theta$

↳ area under the curve



$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_x(\theta)\, d\theta = F_x(b) - F_x(a)$$



Expected value $\mathbb{E}(X)$ = mean $\mu_x = \int_{-\infty}^{\infty} x f_x(x)\, dx$ = 1st raw moment ↓ relative to zero

Central moment → relative to the mean μ_x

⇒ 1st central moment = 0

⇒ 2nd central moment = variance = $\mathbb{E}[(X - \mu_x)(X - \mu_x)]$

$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

⇒ k^{th} central moment = $\mathbb{E}[(X - \mu_x)^k]$

Two random variables X and Y are independent if

$$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$$

Covariance of X and Y is defined as

$$\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

if X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$

$\Rightarrow$ covariance $= 0$ $\Rightarrow$ X and Y are uncorrelated.

claim: $\mathbb{E}\left[(X-\mu_X)(Y-\mu_X)\right] = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$

Proof: $(X-\mu_X)(Y-\mu_Y) = XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y$

$$\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}(XY) - \mu_X\,\mathbb{E}(Y) - \mu_Y\,\mathbb{E}(X) + \mu_X\mu_Y$$

By definition $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) - \mathbb{E}(Y)\,\mathbb{E}(X) + \mathbb{E}(X)\,\mathbb{E}(Y)$$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

Joint probability density $\rightarrow$ describes the $\mathbb{P}(X=\theta_1)$ and $\mathbb{P}(Y=\theta_2)$ simultaneously

$\rightarrow$ $\mathbb{P}(X=\theta_1, \text{ AND } Y=\theta_2)$

Example: Consider two random variables X and Y that has a joint Gaussian distribution

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}(\mu, \Sigma) = \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}\right)$$

Two important concepts

↳ Marginalization $\Rightarrow$ we can extract partial information

$$X \sim \mathcal{N}(\mu_X, \Sigma_{XX}) \quad , \quad Y \sim \mathcal{N}(\mu_Y, \Sigma_{YY})$$

↳ Conditioning $\Rightarrow$ conditional distribution of X, given Y

$$\mathbb{P}(X|\hat{Y}) \sim \mathcal{N}\left(\mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(\hat{Y}-\mu_Y),\ \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)$$

Joint Gaussian distribution of N random variables

$$\mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}, \begin{bmatrix} k(x_1,x_1) & \cdots & k(x_1,x_N) \\ \vdots & \ddots & \vdots \\ k(x_n,x_1) & \cdots & k(x_n,x_n) \end{bmatrix}\right)$$

Random variable – single uncertain quantity

Random signal (random process / stochastic process) is a collection of random variables indexed by time

random signal = $X(t)$

at some time t, $X(t)$ is a random variable.

For a random signal $X(t)$, can we predict $X(t_2)$ if we know $X(t_1)$?

i.e. are $X(t_1)$ and $X(t_2)$ correlated for $t_1 \neq t_2$?

Autocorrelation function for a random signal $X(t)$ is defined as

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)\, X(t_2)]$$

Autocovariance:

$$C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X(t_1))\,(X(t_2) - \mu_X(t_2))]$$

⇒ Autocorrelation is relative to zero, autocovariance is relative to its mean

⇒ $R_X(t_1, t_2) = C_X(t_1, t_2)$ if $X(t)$ has zero mean.

Stationary random process ⇒ statistical properties do not change with time.

↳ PDF can be shifted in time, without affecting its statistical properties.

→ Strict sense stationary process

→ All moments remain the same, when shifted in time

→ Wide-sense stationary process

→ 1st and 2nd moments remain the same, when shifted in time.

$$\mu_X(t) = \mu_X(t+\tau) = \mu_X, \text{ for all } t, \tau \in \mathbb{R}$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2, 0) \text{ for all } t_1, t_2 \in \mathbb{R}$$

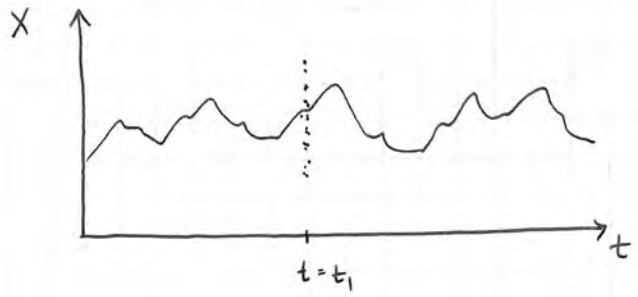
$R_X(\tau) = R_X(t - \tau)$ is the autocorrelation at lag τ.

$$R_X(\tau) = \mathbb{E}[X(t)\, X(t+\tau)]$$

A stationary process is ergodic, if time averages = ensemble averages

Given a random signal $X(t)$.

For each time t, $X(t)$ is a random variable



$$\mu_X(t) = \mu_X = \mathbb{E}(X(t))$$

Ensemble := different realizations of the process at a fixed time.

$$\bar{X}_T = \frac{1}{T}\int_0^T X(t)\,dt$$

The process is ergodic if

$$\lim_{T\to\infty} \bar{X}_T = \mathbb{E}(X(t) \quad \text{with probability 1.}$$

Stationary process $\Rightarrow$ statistics don't change over time

Ergodicity $\Rightarrow$ those statistics can be estimated from a single long trajectory.

Ensemble autocorrelation

$$R_X(\tau) = \mathbb{E}[X(t)X(t-\tau)]$$

Time-averaged auto correlation

$$R_T(\tau) = \frac{1}{T}\int_0^T X(t)\,X(t-\tau)\,dt \quad \leftarrow \text{we can measure in practice}$$

if the process is ergodic, then $R_X(\tau) = R_T(\tau)$ as $T\to\infty$, w.p. 1

$$R_X(\tau) = \lim_{T\to\infty} R_T(\tau) \text{ w.p. 1 } \forall\,\tau$$

Spectral density ⇒ power contained at a specific frequency ω.

↳ move to the frequency domain

Power spectral density = Fourier transform of the autocorrelation

$$S_X(\omega) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

Conversely

$$R_X(\tau) = \mathcal{F}^{-1}[S_X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega\tau}\, d\tau$$

Total average power = mean squared value

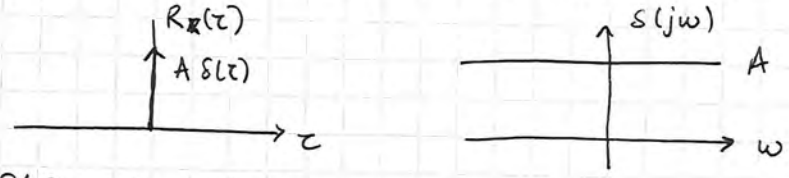
$$P = \int_{-\infty}^{\infty} S_X(\omega)\, d\omega = R_X(0) = \mathbb{E}[(X(t))^2]$$

→ if auto-correlation is long, PSD is concentrated at a few frequencies

⇒ slowly decaying process

→ if auto-correlation is short, PSD is spread out and flat

White noise has a constant PSD



$\delta(\tau)$ - direc function

Random State-space model

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

$$y = Cx$$

x_0 and u are uncertain, therefore x, and y are also uncertain.

Uncertain Solution:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(t)\, d\tau$$

$$y(t) = C x(t)$$

$u(t)$, $y(t)$, $x(t)$ are random signals, and we want to characterize the probability distribution of possible outcomes

Expectation operator $\mathbb{E}$ is a linear operator satisfying

- $\mathbb{E}(x + c) = \mathbb{E}(x) + c$
- $\mathbb{E}(x_1 + x_2) = \mathbb{E}(x_1) + \mathbb{E}(x_2)$
- $\mathbb{E}(a x_1) = a\,\mathbb{E}(x_1)$

Define the following quantities using the $\mathbb{E}$ operator

- Mean: $\mu_x(t) = \mathbb{E}(x(t))$
- Variance $= \mathbb{E}\left[(x(t) - \mu_x(t))^2\right]$
- Co-variance: $\mathrm{cov}[x_1(t), x_2(t)] = \mathbb{E}\left[(x_1 - \mu_{x_1}(t))(x_2 - \mu_{x_2}(t))^T\right]$

Mean of the random process $\mu_x(t) = \mathbb{E}[x(t)]$

$$\mathbb{E}[x(t)] = \mathbb{E}\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau\right]$$

$$= \mathbb{E}\left[e^{At}x_0\right] + \mathbb{E}\left[\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau\right]$$

$$= e^{At}\mathbb{E}[x_0] + \int_0^t e^{A(t-\tau)} B\, \mathbb{E}[u(\tau)]\, d\tau$$

$$\mu_x(t) = e^{At}\mu_{x_0} + \int e^{A(t-\tau)} B\, \mu_u(\tau)\, d\tau$$

$$\mathbb{E}[y(t)] = C\,\mathbb{E}[x(t)]$$

$$\mu_y = C\,\mu_x$$

Given the system describing the random process

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Mean of the random process is given using the expectation operator

$$\mathbb{E}[\dot{x}] = A\,\mathbb{E}[x] + B\,\mathbb{E}[u]$$

$$\mathbb{E}[y] = C\,\mathbb{E}[x]$$

defining $\mu_x := \mathbb{E}[x]$, $\mu_u := \mathbb{E}[u]$, $\mu_y := \mathbb{E}[y]$

$$\Rightarrow \quad \dot{\mu}_x = A\mu_x + B\mu_u \qquad \text{with } \mu_{x_0} = \mathbb{E}[x_0]$$

$$\mu_y = C\mu_x$$

LTI solution

$$\mu_x(t) = e^{At}\mu_{x_0} + \int_0^t e^{A(t-\tau)} B\mu_u(\tau)\,d\tau$$

which is the same as the expectation operator applied on the LTI solution of the random process

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B\mu_u(\tau)\,d\tau$$

$$\mathbb{E}[x(t)] = \mathbb{E}[e^{At}x_0] + \mathbb{E}\left[\int_0^t e^{A(t-\tau)} B\mu_u(\tau)\,d\tau\right]$$

$$\mathbb{E}[x(t)] = e^{At}\,\mathbb{E}[x_0] + \int_0^t e^{A(t-\tau)} B\,\mathbb{E}[\mu_u(\tau)]\,d\tau$$

$$\Rightarrow \quad \mu_x(t) = e^{At}\mu_{x_0} + \int_0^t e^{A(t-\tau)} B\mu_u(\tau)\,d\tau$$

Note: deterministic models are found in the limit of

$$\text{Variance}[x] \to 0$$

Now that we have a model for the mean, let's try to model the covariance

↳ First let's define the covariance.

If we have two random signals $X_1(t)$ and $X_2(t)$

$$\text{Var}[X_1(t)] = \mathbb{E}[(X_1(t) - \mu_{X_1}(t))(X_1(t) - \mu_{X_1}(t))]$$

$$\text{Var}[X_2(t)] = \mathbb{E}[(X_2(t) - \mu_{X_2}(t))(X_2(t) - \mu_{X_2}(t))]$$

Simplify and drop (t) notation

$$\text{Var}[X_i] = \mathbb{E}[(X_i - \mu_{X_i})(X_i - \mu_{X_i})]$$

Covariance measures relation between X_1 and X_2

$$\text{cov}[X_1, X_2] = \mathbb{E}[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})] = \mathbb{E}[(X_2 - \mu_{X_2})(X_1 - \mu_{X_1})]^T = \text{cov}[X_2, X_1]$$

if $X(t) \in \mathbb{R}^n$, then we have a covariance matrix C_X

$$C_X := \begin{bmatrix} \mathbb{E}[(X_1 - \mu_{X_1})(X_1 - \mu_{X_1})] & \mathbb{E}[(X_1 - \mu_{X_1})(X_2 - \mu_{X_1}) & \cdots & \mathbb{E}[(X_1 - \mu_{X_1})(X_n - \mu_{X_n})] \\ \mathbb{E}[(X_2 - \mu_{X_2})(X_1 - \mu_{X_1})] & \mathbb{E}[(X_2 - \mu_{X_2})(X_2 - \mu_{X_2})] & \cdots & \mathbb{E}[(X_2 - \mu_{X_2})(X_n - \mu_{X_n})] \\ \vdots & & & \\ \mathbb{E}[(X_n - \mu_{X_n})(X_1 - \mu_{X_1})] & \cdots & & \mathbb{E}[(X_n - \mu_n)(X_n - \mu_n)] \end{bmatrix}$$

$$C_X = \begin{bmatrix} \text{Var}[X_1] & \text{cov}[X_1, X_2] & \cdots & \text{cov}[X_1, X_n] \\ \text{cov}[X_2, X_1] & \text{Var}[X_2] & \cdots & \text{cov}[X_2, X_n] \\ \vdots & & & \\ \text{cov}[X_n, X_1] & \text{cov}[X_n, X_n] & \cdots & \text{Var}[X_n] \end{bmatrix}$$

Vectorial representation:

$$C_X = \mathbb{E}[(X(t) - \mu_X(t))(X(t) - \mu_X(t))^T]$$

because $\text{cov}[X_i, X_j] = \text{cov}[X_j, X_i]$, Covariance matrix C_X is symmetric

Given a random process

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

we saw that, the mean model is given by

$$\dot{\mu}_x = A\mu_x + B\mu_u$$

$$\mu_y = C\mu_x$$

Define the deviation from the mean

$$\bar{x} = x - \mu_x \;,\; \bar{u} = u - \mu_u \;,\; \bar{y} = y - \mu_y$$

$$\dot{\bar{x}} = \dot{x} - \dot{\mu}_x = Ax + Bu - A\mu_x - B\mu_u$$

$$\Rightarrow \dot{x} - \dot{\mu}_x = A(x - \mu_x) + B(u - \mu_u)$$

$$y - \mu_y = Cx - C\mu_x \Rightarrow y - \mu_y = C(x - \mu_x)$$

$\Rightarrow$ Random state space model about the mean

$$\dot{x} - \dot{\mu}_x = A(x - \mu_x) + B(u - \mu_u)$$

$$y - \mu_y = C(x - \mu_x)$$

Let's go back to the covariance matrix in vector form

$$C_x = \mathbb{E}[(x - \mu_x)(x - \mu_x)^T]$$

Covariance update equation

$$\dot{C}_x = \mathbb{E}[(\dot{x} - \dot{\mu}_x)(x - \mu_x)^T] + \mathbb{E}[(x - \mu_x)(\dot{x} - \dot{\mu}_x)^T]$$

$$= A(x - \mu_x) + B(u - \mu_u)$$

$$\dot{C}_x = \mathbb{E}[(A(x - \mu_x) + B(u - \mu_u))(x - \mu_x)^T] + \mathbb{E}[(x - \mu_x)(A(x - \mu_x) + B(u - \mu_u))^T]$$

$$\dot{C}_x = \mathbb{E}[A(x - \mu_x)(x - \mu_x)^T] + \mathbb{E}[B(u - \mu_u)(x - \mu_x)^T] + \mathbb{E}[(x - \mu_x)(x - \mu_x)^T A^T] + \mathbb{E}[(x - \mu_x)(u - \mu_u)^T B^T]$$

$$\dot{C}_x = A\,\mathbb{E}[(x - \mu_x)(x - \mu_x)^T] + B\,\mathbb{E}[(u - \mu_u)(x - \mu_x)^T] + \underbrace{\mathbb{E}[(x - \mu_x)(x - \mu_x)^T]}_{= C_x} A^T + \mathbb{E}[(x - \mu_x)(u - \mu_u)^T] B^T$$

$$\dot{C}_x = A C_x + C_x A^T + B \underbrace{\mathbb{E}[(u-\mu_u)(x-\mu_x)^T]}_{\text{what is the covariance between } x \text{ and } u?} + \mathbb{E}[(x-\mu_x)(u-\mu_u)^T] B^T$$

To understand this, let's look at the LTI solution of the random state-space model about the mean.

$$\dot{x} - \dot{\mu}_x = A(x-\mu_x) + B(u-\mu_u)$$

LTI solution:

$$x(t) - \mu_x(t) = e^{At}(x_0 - \mu_{x_0}) + \int_0^t e^{A(t-\tau)} B(u(\tau) - \mu_u(\tau))\, d\tau$$

→ Substitute into $\mathbb{E}[(x-\mu_x)(u-\mu_u)^T]B^T$ to compute the covariance between x and u.

$$\mathbb{E}[(x-\mu_x)(u-\mu_x)^T]B^T = \mathbb{E}\left[\left(e^{At}(x_0-\mu_{x_0}) + \int_0^t e^{A(t-\tau)} B(u(\tau)-\mu_u(\tau))\, d\tau\right)(u-\mu_x)^T\right] B^T$$

$$\Rightarrow \mathbb{E}[e^{At}(x_0-\mu_{x_0})(u-\mu_u)]B^T + \mathbb{E}\left[\int_0^t e^{A(t-\tau)} B(u(\tau)-\mu_u(\tau))(u-\mu_u)^T\right] B^T$$

$$\Rightarrow e^{At}\underbrace{\mathbb{E}[(x_0-\mu_{x_0})(u-\mu_u)]}_{=0,\ \text{because of causality.}} B^T + \mathbb{E}\left[\int_0^t e^{A(t-\tau)} B(u(\tau)-\mu_u(\tau))(u-\mu_u)^T\right] B^T$$

i.e., inputs $u(t)$ cannot influence states at time $t=0$

$$\Rightarrow \mathbb{E}[(x-\mu_x)(u-\mu_u)^T]B^T = \mathbb{E}\left[\int_0^t e^{A(t-\tau)} B(u(\tau)-\mu_u(\tau))(u-\mu_u)^T\right] B^T$$

$$= \int_0^t e^{A(t-\tau)} B \underbrace{\mathbb{E}[(u(\tau)-\mu_u(\tau))(u(t)-\mu_u(t))]}_{\text{Auto covariance of the input } u(t)} d\tau\, B^T$$

→ Assume randomness in input u is modeled as white noise

For a white noise, the autocovariance is given by



i.e. Autocovariance $= 0$, $\forall \tau \neq t$

at $t = \tau$, Autocovariance $= \delta(0)\, Q_u$, $Q_u \succ 0$

integral of dirac delta $f(\tau) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau = f(t)$

but our integral is from $0-t$, $\int_0^t f(\tau)\, \delta(t-\tau)\, d\tau = \frac{1}{2} f(t)$

$f(\tau) = e^{A(t-\tau)} B Q_u B^T$, at $\tau = t$, $e^{A(t-t)} = e^{A0} = I$

Modelling the randomness in the input as a white noise allows us to simplify the term

$$\int_0^t e^{A(t-\tau)} B \mathbb{E}\left[(u(\tau) - \mu_u(\tau))(u(t) - \mu_u(t))^T\right] B^T = \frac{1}{2} B Q_u B^T$$

Back to our covariance update equation

$$\dot{C}_x = A C_x + C_x A^T + \underbrace{B \mathbb{E}[(u-\mu_u)(x-\mu_x)^T]}_{= \left(\frac{1}{2} B Q_u B^T\right)^T} + \underbrace{\mathbb{E}[(x-\mu_x)(u-\mu_u)^T] B^T}_{= \frac{1}{2} B Q_u B^T}$$

Since Q_u is symmetric, $\left(\frac{1}{2} B Q_u B^T\right)^T = \frac{1}{2} B Q_u B^T$

$$\Rightarrow \dot{C}_x = A C_x + C_x A^T + B Q_u B^T$$

Covariance of y, $C_y = \mathbb{E}[(y-\mu_y)(y-\mu_y)^T]$

$$C_y = \mathbb{E}[C(x-\mu_x)(x-\mu_x)^T C^T] = C \underbrace{\mathbb{E}[(x-\mu_x)(x-\mu_x)^T]}_{= C_x} C^T$$

$$\Rightarrow C_y = C C_x C^T$$

$\Rightarrow$ Covariance model is thus given by:

$$\dot{C}_x = A C_x + C_x A^T + B Q_u B^T$$

$$C_y = C C_x C^T$$

To summarize:

Assume

→ input can be modelled as a white noise, with auto covariance

$$\mathbb{E}[(u-\mu_u)(u-\mu_u)^T] = \delta(0) Q_u \,, \quad Q_u \succ 0$$

with known variance $Q_u \succ 0$

→ known covariance for the initial state

$$\mathbb{E}[(x_0 - \mu_{x0})(x_0 - \mu_{x0})^T] = C_{x0}$$

→ known means $\mathbb{E}[u(t)] = \mu_u(t)$ and $\mathbb{E}[x_0] = \mu_{x0}$

The means are given by:

$$\dot{\mu}_x = A\mu_x + B\mu_u \quad \text{with} \quad \mu_{x_0} = \mu_x(0)$$

$$\mu_y = C\mu_x$$

and the covariance is given by:

$$\dot{C}_x = A C_x + C_x A^T + B Q_u B^T \quad \text{with} \quad C_x(0) = C_{x_0}$$

$$C_y = C C_x C^T$$

We now know how both the mean and the covariance changes for our random process

Classical linear least squares estimation

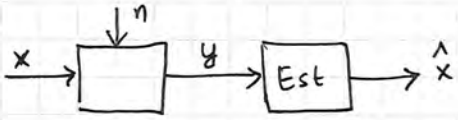
$x \rightarrow$ true state

$y \rightarrow$ noisy measurement

$$y = x + n$$

Goal: Build a linear estimator to filter out the noise

$$\hat{x} = Ly$$



we want $x = \hat{x}$ ideally

minimize the mean squared error

$$\min_L J(L) = \mathbb{E}\left[(\hat{x}-x)(\hat{x}-x)^T\right] \quad \leftarrow \text{minimize error co-variance}$$

Simple case where x is zero mean

State co-variance $Q = \mathbb{E}[xx^T]$

n – zero mean noise, $\mathbb{E}[nn^T] = R$

Assume x and n are uncorrelated $\mathbb{E}[xn^T] = 0$

$$\min_L \mathbb{E}\left[(Ly - x)(Ly - x)^T\right]$$

$$y = x + n$$

$$\min_L \mathbb{E}\left[(Lx + Ln - x)(Lx + Ln - x)^T\right]$$

$$\Rightarrow \min_L \mathbb{E}\left[((L-I)x + Ln)((L-I)x + Ln)^T\right]$$

$$= \mathbb{E}\left[(L-I)xx^T(L-I)^T + Lnn^TL^T + (L-I)xn^TL + Lnx(L-I)\right]$$

$$= \mathbb{E}\left[(L-I)xx^T(L-I)^T\right] + \mathbb{E}\left[Lnn^TL^T\right] + 2\,\mathbb{E}\left[(L-I)xn^TL\right]$$

$$= (L-I)\underbrace{\mathbb{E}[xx^T]}_{=Q}(L-I)^T + L\underbrace{\mathbb{E}[nn^T]}_{=R}L^T + 2(L-I)\underbrace{\mathbb{E}[xn^T]}_{=0}L$$

$$= (L-I)Q(L-I)^T + LRL^T$$

$$\min_{L} \quad (L-I)\,Q\,(L-I)^T + L\,R\,L^T$$

derivative w.r.t. L (matrix calculus rule)

$$\frac{\partial}{\partial L}\,\mathrm{tr}\left[(L-I)\,Q\,(L-I)^T + L\,R\,L^T\right] = 2\,(L-I)\,Q + 2\,L\,R = 0$$

↙ set derivative = 0 for minimum

$$(L-I)\,Q + L\,R = 0$$

$$L\,Q - Q + L\,R = 0$$

$$L\,(Q+R) = Q$$

$$L = Q\,(Q+R)^{-1} = \text{optimal linear least squares gain}$$

Intuition

High state uncertainty $Q \gg R$

↳ trust measurement more

$$L \approx \frac{Q}{Q} = I$$

$$\hat{x} = y$$

High measurement noise $R \gg Q$

↳ trust the prior mean more / model

$$\Rightarrow \quad L \approx 0$$

Estimation error covariance. $e = \hat{x} - x$

$$\mathbb{E}[e\,e^T] = \mathbb{E}\left[(\hat{x}-x)(\hat{x}-x)^T\right]$$

$$= (L-I)\,Q\,(L-I)^T + L\,R\,L^T$$

Substitute $L = Q\,(Q+R)^{-1}$

$\Rightarrow$ error covariance $= (Q^{-1} + R^{-1})^{-1}$

$$\dot{x} = Ax + Bu + Gw$$

$$y = Cx + v$$

w - process noise - zero mean Gaussian white noise with covariance $E[ww^T] = Q$

v - measurement noise - zero mean Gaussian white noise with covariance $E[vv^T] = R$

process and measurement noise uncorrelated : i.e., $E[wv^T] = 0$

We want to design a Luenberger observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Estimation error $e = x - \hat{x}$ (random signal)

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu + Gw - A\hat{x} - Bu - L(y - C\hat{x})$$

$$= Ax + Bu + Gw - A\hat{x} - Bu - L(Cx + v - C\hat{x})$$

$$= A(x - \hat{x}) - LC(x - \hat{x}) + Gw - Lv$$

$$= (A - LC)e + Gw - Lv$$

Random error dynamics

$$\dot{e} = (A - LC)e + Gw - Lv \quad , \quad e(t),\ w(t),\ v(t) \text{ are random signals}$$

Mean model :

$$\dot{\mu}_e = (A - LC)\mu_e \quad \text{(since } w \text{ \& } L \text{ are zero mean noises)}$$

Covariance model

define $P := E[ee^T]$

$$\dot{P} = (A - LC)P + P(A - LC)^T + \underbrace{E[e(w^TG^T - v^TL^T)] + E[e(w^TG^T - v^TL^T)]^T}$$

$$= GQG^T + LRL^T$$

Since w & v are Gaussian white noises and are uncorrelated

$$\dot{P} = (A - LC)P + P(A - LC)^T + GQG^T + LRL^T$$

Minimize error covariance

$$\min_{L} J(L) = E[e e^T] = \text{trace}(P)$$

$$\frac{\partial}{\partial L} \text{tr}(P) = 0 \rightarrow L^* = \text{optimal gain} = \text{Kalman gain}$$

optimal gain will depend on the process covariance Q and measurement covariance R.

→ $Q >> R$ ⇒ high model / state uncertainty
↳ trust measurement more

→ $R >> Q$ ⇒ high measurement noise
↳ trust the model more.

Kalman filter in Continuous time

Plant is given by a random process

$$\dot{x} = Ax + Bu + Gw$$
$$y = Cx + v$$

x, y, w, v are random signals originating from the randomness in w and v.

w – Process noise / disturbance
↳ Zero mean white Gaussian signal with auto covariance $\mathbb{E}[w w^T] = Q$

v – measurement noise
↳ Zero mean white Gaussian signal with auto covariance $\mathbb{E}[v v^T] = R$

Process & measurement noise are uncorrelated
i.e. $\mathbb{E}[w v^T] = 0$

Recall Luenberger observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

NB! $\hat{x}$ is also random since it depends on the random meas. y.

Estimation error $e = x - \hat{x}$

error dynamics $= \dot{e} = \dot{x} - \dot{\hat{x}}$

$$\dot{e} = Ax + Bu + Gw - A\hat{x} - Bu - L(y - C\hat{x})$$
$$= A(x - \hat{x}) + Gw - L(Cx + v - C\hat{x})$$
$$= A(x - \hat{x}) + Gw - LC(x - \hat{x}) - Lv$$
$$= Ae - LCe + Gw - Lv$$

$$\Rightarrow \quad \dot{e} = (A - LC)e + Gw - Lv$$

↳ Treat this as a random S.S. model and write the mean & covariance model

Mean model

$$\mu_e = (A - LC)\mu_e \quad \text{if } \mu_{e_0} = 0 \Rightarrow \text{estimate is unbiased.}$$

i.e. $\mathbb{E}[\hat{x}] = \mathbb{E}[x]$

Covariance model

Let $P := \mathbb{E}[e e^T]$ be the error covariance

$$\dot{P} = (A - LC)P + P(A-LC)^T + GQG^T + LRL^T$$

How to design observer gain L?

↳ choose L such that the error covariance decreases at the fastest possible rate

$$\min_L \dot{P} \quad \Rightarrow$$

⇒ take derivative of $\dot{P}$ w.r.t L and set it to zero

$$\frac{\partial}{\partial L} tr(\dot{P}) = 0 \qquad \text{(Necessary condition of optimality)}$$

Matrix differentiation rule

→ $\frac{\partial}{\partial X} tr(XA) = A^T$

→ $\frac{\partial}{\partial X} tr(AX^T) = A$

→ $\frac{\partial}{\partial X} tr(XBX^T) = XB^T + XB = 2XB$ if B is symmetric

$$\frac{\partial}{\partial L} tr(\dot{P}) = \frac{\partial}{\partial L} tr\left[(A-LC)P + P(A-LC)^T + GQG^T + LRL^T\right]$$

$$= \frac{\partial}{\partial L} tr\left[AP - LCP + PA^T - PC^TL^T + GQG^T + LRL^T\right]$$

$$= \underbrace{-P^TC^T - PC^T}_{= -2PC^T \text{ since } P^T \text{ is symmetric}} + \underbrace{LR^T + LR}_{= 2LR \text{ since } R \text{ is symmetric}}$$

$$= -2PC^T + 2LR = 0$$

$$\Rightarrow \quad LR = PC^T$$

$$L = PC^TR^{-1} \quad \leftarrow \text{Kalman gain}$$

Second-order sufficient condition for optimality

$$\frac{\partial^2}{\partial L^2} tr(\dot{P}) > 0$$

Let's verify this $\frac{\partial^2}{\partial L^2} tr(\dot{P}) = R > 0 \Rightarrow L = PC^TR^{-1}$ is the optimum.

error covariance under optimal L is then given by the following matrix Riccati Equation

$$\dot{P} = AP + PA^T + GQG^T - PC^T R^{-1} CP$$

Proof:

$$\dot{P} = (A - LC)P + P(A - LC)^T + GQG^T + LRL^T$$

$$= AP - LCP + PA^T - PC^T L^T + GQG^T + LRL^T$$

$$= AP + PA^T + GQG^T - LCP - PC^T L^T + LRL^T$$

Substitute $L = PC^T R^{-1}$

$$\dot{P} = AP + PA^T + GQG^T - PC^T R^{-1} CP - PC^T R^{-1T} C^T P^T + PC^T R^{-1} R R^{-1T} CP^T$$

$$= AP + PA^T + GQG^T - PC^T R^{-1} CP - PC^T R^{-1T} CP^T + PC^T R^{-1} R^{-1T} CP$$

$$\Rightarrow \dot{P} = AP + PA^T + GQG^T - PC^T R^{-1} CP$$

$$\dot{\hat{x}} = A\hat{x} + Bu + PC^T R^{-1}(y - C\hat{x})$$

Assuming stationary process, covariance will converge to a steady state value.

$$\lim_{t \to \infty} P(t) = P_\infty$$

At steady-state $\dot{P} = 0$

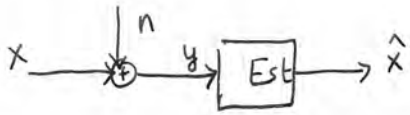
$$\Rightarrow \quad AP_\infty + P_\infty A^T + GQG^T - P_\infty C^T R^{-1} CP_\infty = 0$$

$$L_\infty = P_\infty C^T R^{-1} \leftarrow \text{optimal stationary observer gain}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L_\infty(y - C\hat{x})$$

Discrete-time Kalman filter

Consider a simple process corrupted by noise n



We want to estimate $\hat{x}$ from the noisy measurement y.

assume:
noise has zero mean with covariance $\mathbb{E}[nn^T] = R$

minimum variance unbiased estimator (MVUE)

minimizes the mean squared error

$\hat{x} = \arg\min \mathbb{E}[\|\hat{x} - x\|^2]$, also denoted by

$\min \mathbb{E}[\|\hat{x} - x\|^2 \mid y]$

minimize the mean squared error, given the meas. y.

$\mathbb{E}[\|\hat{x} - x\|^2] = \text{trace}\, \mathbb{E}[(\hat{x} - x)(\hat{x} - x)^T]$

Kalman filter is a minimum variance of error filter.

Discrete time plant-model

$$x_{k+1} = A_d x_k + B_d u_k + w_k$$
$$y_k = C x_k + v_k$$

w_k – process noise / disturbance with zero mean white Gaussian signal with covariance $\mathbb{E}[w_k w_k^T] = Q_d$

v_k – measurement noise – zero mean white Gaussian signal with covariance $\mathbb{E}[v_k v_k^T] = R_d$

assume: process and measurement noise are uncorrelated.

i.e. $\mathbb{E}[w_k v_k^T] = 0$

Note! A, B, C, Q and R here are not the same as the A, B, C, Q and R in continuous time!

in discrete time we consider

A_d, B_d, C_d, Q_d, R_d, but simply drop the subscript d.

Continuous time to discrete-time conversion. with sampling time T

$$A_d = e^{AT}, \quad B_d = \int_0^T e^{A\alpha} d\alpha \, B = A^{-1}(A_d - I)B$$

$$C_d = C, \quad Q_d = \int_0^T e^{A\alpha} G Q G^T e^{A\alpha} d\alpha, \quad R_d = R/T$$

Drop the subscript d, and express our discrete time system as:

$$x_{k+1} = A x_k + B u_k + G w_k \quad (\text{assume } G = I)$$

$$y_k = C x_k + v_k$$

with $E[w_k w_k^T] = Q$, $E[v_k v_k^T] = R$, $E[w_k v_k^T] = 0$

Discrete-time observer

↳ Estimation proceeds in two distinct steps

→ Prediction step / A-priori – $\hat{x}_{k|k-1}$

↳ best guess for $\hat{x}_k$ at time step $k-1$, prior to the incorporation of y_k using the deterministic model

$$\hat{x}_{k|k-1} = A \hat{x}_{k-1|k-1} + B u_{k-1}$$

→ Update step / A-posteriori – $\hat{x}_{k|k}$

↳ best guess for $\hat{x}_k$ after observing the measurement y_k

↳ linear blend of the model prediction and the new measurement

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C\hat{x}_{k|k-1})$$

Estimation error : $e_k = x_k - \hat{x}_{k|k}$

error covariance $P_{k|k} := E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T]$

mean squared error = $tr(P_{k|k})$

choose L_k such that we min $tr(P_{k|k})$

$$L_k = \min_L tr(P_{k|k}) \quad \rightarrow \text{Kalman filter.}$$

At time step k, the true plant evolves as:

$$x_k = A x_{k-1} + B u_{k-1} + w_{k-1}$$

But our a-priori prediction reads as:

$$\hat{x}_{k|k-1} = A \hat{x}_{k-1|k-1} + B u_{k-1}$$

The a-priori error at this stage:

$$\begin{aligned} e_{k|k-1} &= x_k - \hat{x}_{k|k-1} \\ &= A x_{k-1} + B u_{k-1} + w_{k-1} - A \hat{x}_{k-1|k-1} - B u_{k-1} \\ &= A \underbrace{(x_{k-1} - \hat{x}_{k-1|k-1})}_{:= e_{k-1}} + w_{k-1} \end{aligned}$$

$$e_{k|k-1} = A e_{k-1} + w_{k-1}$$

a-priori error co-variance $:= P_{k|k-1}$

$$\begin{aligned} P_{k|k-1} &= \mathbb{E}\left[e_{k|k-1} \, e_{k|k-1}^T \right] \\ &= \mathbb{E}\left[(A e_{k-1} + w_{k-1}) (A e_{k-1} + w_{k-1})^T \right] \\ &= \mathbb{E}\left[A e_{k-1} e_{k-1}^T A^T \right] + \mathbb{E}\left[w_{k-1} w_{k-1}^T \right] + \underbrace{\mathbb{E}\left[e_{k-1} w_{k-1}^T \right]}_{= 0} \quad \text{additional terms} \\ &= \mathbb{E}\left[A e_{k-1} e_{k-1}^T A^T \right] + \mathbb{E}\left[w_{k-1} w_{k-1}^T \right] \\ &= A \, \mathbb{E}\left[e_{k-1} e_{k-1}^T \right] A^T + \mathbb{E}\left[w_{k-1} w_{k-1}^T \right] \\ &= A P_{k-1|k-1} A^T + Q \end{aligned}$$

$$\Rightarrow \quad P_{k|k-1} = A P_{k-1|k-1} A^T + Q$$

So, After the prediction / a-priori step we have

$$\hat{x}_{k|k-1} = A \hat{x}_{k-1|k-1} + B u_{k-1}$$

with apriori error $e_{k|k-1} = A e_{k-1} + w_{k-1}$ with covariance

$$P_{k|k-1} = A P_{k-1|k-1} A^T + Q.$$

Update / a posteriori step

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L\left(y_k - C\hat{x}_{k|k-1}\right)$$

observed measurement $y_k = Cx_k + v_k$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L\left(Cx_k + v_k - C\hat{x}_{k|k-1}\right)$$

$$= \hat{x}_{k|k-1} + LC\underbrace{\left(x_k - \hat{x}_{k|k-1}\right)}_{:= e_{k|k-1}} + Lv_k$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + LC\,e_{k|k-1} + Lv_k$$

estimation error after the update / a posteriori error

$$e_k = x_k - \hat{x}_{k|k}$$

$$= \underbrace{x_k - \hat{x}_{k|k-1}}_{= e_{k|k-1}} - LC\,e_{k|k-1} + Lv_k$$

$$e_k = e_{k|k-1} - LC\,e_{k|k-1} + Lv_k$$

$$\Rightarrow e_k = (I - LC)\,e_{k|k-1} + Lv_k$$

Estimation error covariance $\mathbb{E}[e_k e_k^T] = P_{k|k}$

$$P_{k|k} = \mathbb{E}[e_k e_k^T]$$

$$= \mathbb{E}\left[\left((I-LC)e_{k|k-1} + Lv_k\right)\left((I-LC)e_{k|k-1} + Lv_k\right)^T\right]$$

$$= \mathbb{E}\left[(I-LC)\,e_{k|k-1}\,e_{k|k-1}^T\,(I-LC)^T\right] + \mathbb{E}\left[Lv_k v_k^T L^T\right]$$

$$+ \underbrace{\text{additional terms}}_{=0}$$

· since $e_{k|k-1}$ and v_k are uncorrelated.

$$P_{k|k} = \mathbb{E}\left[(I-LC)\,e_{k|k-1}\,e_{k|k-1}^T\,(I-LC)^T\right] + \mathbb{E}\left[Lv_k v_k^T L^T\right]$$

$$= (I-LC)\underbrace{\mathbb{E}\left[e_{k|k-1}\,e_{k|k-1}^T\right]}_{= P_{k|k-1}}(I-LC)^T + L\underbrace{\mathbb{E}\left[v_k v_k^T\right]}_{= R}L^T$$

$$P_{k|k} = (I-LC)\,P_{k|k-1}\,(I-LC)^T + LRL^T$$

Kalman gain = choose L such that we minimize

$$\operatorname{trace}(P_{k|k})$$

$$\min_L \operatorname{tr}(P_{k|k})$$

$\frac{\partial}{\partial L}\operatorname{tr}(P_{k|k}) = 0$ → necessary condition of optimality

$$\frac{\partial}{\partial L} tr(P_{k|k}) = \frac{\partial}{\partial L} tr\left[(I - LC) P_{k|k-1} (I - LC)^T + L R L^T \right]$$

using matrix differentiation rules

$$\frac{\partial}{\partial L} tr(P_{k|k}) = -2 (I - LC) P_{k|k-1} C^T + 2 L R = 0$$

$$\Rightarrow (I - LC) P_{k|k-1} C^T + L R = 0$$

$$\Rightarrow P_{k|k-1} C^T - L C P_{k|k-1} C^T + L R = 0$$

$$\Rightarrow P_{k|k-1} C^T - L (C P_{k|k-1} C^T + R) = 0$$

$$\Rightarrow L = P_{k|k-1} C^T (C P_{k|k-1} C^T + R)^{-1} \leftarrow \text{Kalman gain}$$

Discrete-time Kalman filter algorithm

→ Predict / apriori step, starting from $\hat{x}_{0|0} = \mu_{x_0}$, $P_{0|0} = C_{x_0}$

$$\hat{x}_{k|k-1} = A \hat{x}_{k-1|k-1} + B u_{k-1}$$

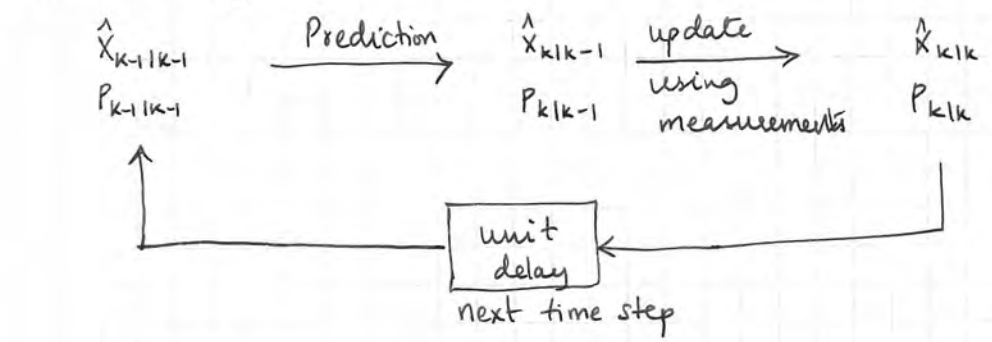
$$P_{k|k-1} = A P_{k-1|k-1} A^T + Q$$

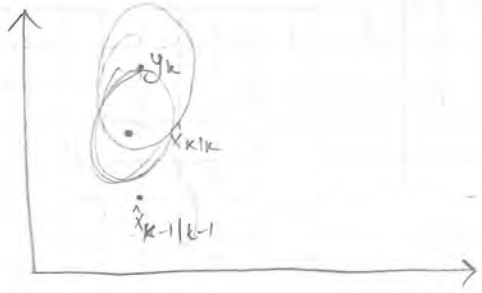
→ Update / aposteriori step

$$L_k = P_{k|k-1} C^T (C P_{k|k-1} C^T + R)^{-1}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - C \hat{x}_{k|k-1})$$

$$P_{k|k} = (I - L_k C) P_{k|k-1} (I - L_k C)^T + L_k R L_k^T$$





Time varying models

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + G(t)\,w(t)$$

$$y(t) = C(t)\,x(t) + v(t)$$

process & measurement noise have zero mean and white with covariance $Q(t)$ and $R(t)$ respectively.

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + L(t)\,(y(t) - C(t)\,\hat{x}(t))$$

$$L(t) = P(t)\,C^T(t)\,R^{-1}(t)$$

$$\dot{P}(t) = A(t)\,P(t) + P(t)\,A(t)^T + G(t)\,Q(t)\,G(t) - P(t)\,C(t)^T R(t)^{-1} C(t)\,P(t)$$

↳ Simulation of the matrix Riccati equation computationally challenging

↳ Better to do in discrete time

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

$$y_k = C_k x_k + v_k$$

process & measurement noise are zero mean white noise with covariance Q_k and R_k respectively.

Discrete-time Kalman filter

→ Predict / apriori step

$$\hat{x}_{k|k-1} = A_k \hat{x}_{k-1|k-1} + B_k u_{k-1}$$

$$P_{k|k-1} = A_k P_{k-1|k-1} A_k^T + Q_k$$

→ Update / a posteriori step

$$L_k = P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - C_k \hat{x}_{k|k-1})$$

$$P_{k|k} = (I - L_k C_k) P_{k|k-1} (I - L_k C_k)^T + L_k R_k L_k^T$$

Which is similar to the LTI case, except for the fact that the matrices $A, B, C, Q,$ & R are time varying.

Extended Kalman filter (preview)

What if our plant is nonlinear?

$$x_{k+1} = f(x_k, u_k) + w_k$$
$$y_k = h(x_k) + v_k$$

Same assumptions about process and measurement noise.

↳ Succesive linearization around the current state.

$$A_k = \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_{k|k},\, u=u_k} \qquad B_k = \left.\frac{\partial f}{\partial u}\right|_{x=\hat{x}_{k|k},\, u=u_k}$$

$$C_k = \left.\frac{\partial h}{\partial x}\right|_{x=\hat{x}_{k|k}}$$

Linearized Time varying model is given by

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$
$$y_k = C_k x_k + v_k$$

and apply the discrete-time Kalman filter as before.

Given a system

$$\dot{x} = Ax + Bu + Gw$$

$$y = Cx + v$$

So far we assumed that the process noise w and measr. noise v are white.

But let's relax this assumption, and consider w to be a coloured noise. (i.e., replace w with d)

$$\dot{x} = Ax + Bu + Gd$$

$$y = Cx + v$$

Let's assume there exists a transfer function $H_d(s)$ which generates the coloured noise d from white noise w with covariance q_w

i.e. $w \rightarrow \boxed{H_d(s)} \rightarrow d$

i.e. the power spectrum of d is given by

$$S_d(\omega) = q_w \left| H_d(j\omega) \right|^2$$

We can realize this transfer function $H_d(s)$ as a state space

$$H_d(s) = C_d \left(sI - A_d \right)^{-1} B_d + D_d$$

i.e., the coloured noise is given by the dynamics

$$\left.\begin{aligned} \dot{x}_d &= A_d x_d + B_d w \\ d &= C_d x_d + D_d w \end{aligned}\right\} \leftarrow \text{disturbance model}$$

if we include this into our model, then the assumptions we made for Kalman filter are valid.

Augment plant model with disturbance model

$$\begin{bmatrix} \dot{x} \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} A & GC_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x \\ x_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} GD_d \\ B_d \end{bmatrix} w$$

Plant model:

$$\dot{x} = Ax + Bu + Gd$$

From the disturbance model $d = C_d x_d + D_d w$

$$\Rightarrow \quad \dot{x} = Ax + Bu + GC_d x_d + GD_d w$$

$$\dot{x}_d = A_d x_d + B_d w$$

write this in matrix form

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_d \end{bmatrix}}_{:=\dot{z}} = \underbrace{\begin{bmatrix} A & GC_d \\ 0 & A_d \end{bmatrix}}_{=A_z} \underbrace{\begin{bmatrix} x \\ x_d \end{bmatrix}}_{:=z} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{:=B_z} u + \underbrace{\begin{bmatrix} GD_d \\ B_d \end{bmatrix}}_{G_z} w$$

$$\Rightarrow \quad \dot{z} = A_z z + B_z u + G_z w$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} z + v$$

Now w and v are white noises, and we can apply the Kalman filter on this augmented system to estimate x and x_d.

If our process has coloured noise d, we need to find a filter $H_d(s)$ that colours white noise into d.

↳ and include this into the system.

In discrete time

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$y_k = Cx_k + v_k$$

We assumed w_k & v_k to be white noise so far.
But let's assume w_k is now a coloured noise.

White noise w_k

$$\mathbb{E}[w_k w_l] = \begin{cases} Q & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

i.e. w_k does not depend on w_{k-1}

If process noise is not white, then let's denote the system as:

$$x_{k+1} = Ax_k + Bu_k + d_k$$

$$y_k = Cx_k + v_k$$

where d_k is a coloured noise, i.e., it is correlated over time.

$\Rightarrow$ d_{k+1} depends on d_k

$$d_{k+1} = Fd_k + w_k \quad \text{with } w_k \text{ being a zero mean white noise}$$

Augment the plant model with the disturbance model

$$x_{k+1} = Ax_k + Bu_k + d_k$$

$$d_{k+1} = Fd_k + w_k$$

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & I \\ 0 & F \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ I \end{bmatrix} w_k$$

$$y_{k1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + v_k$$

↑ Apply the discrete-time Kalman filter on this augmented system

Augmenting the state means, we let the filter learn the noise dynamics, so that we can predict and cancel them.

More generally, the noise model can be expressed as:

$$x_{d_{k+1}} = A_d x_{d_k} + B_d w_k$$

$$d_k = C_d x_{d_k} + D_d w_k$$

Given a discrete-time system with coloured process noise d_k.

$$x_{k+1} = Ax_k + Bu_k + d_k$$

$$y_k = Cx_k + v_k$$

$$\Rightarrow x_{k+1} = Ax_k + Bu_k + C_d x_{d_k} + D_d w_k$$

$$x_{d_{k+1}} = A_d x_{d_k} + B_d w_k$$

$$\Rightarrow \begin{bmatrix} x_{k+1} \\ x_{d_{k+1}} \end{bmatrix} = \begin{bmatrix} A & C_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x_k \\ x_{d_k} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} D_d \\ B_d \end{bmatrix} w_k$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{d_k} \end{bmatrix} + v_k$$

↳ Apply Kalman filter on this augmented system

Given transfer function $H_d(s)$ for the colouration of white noise.

$$w(t) \longrightarrow \boxed{H_d(s)} \longrightarrow d(t)$$

realize continuous time state-space from $H_d(s)$

↳ convert into discrete-time state-space

(or)

Convert $H_d(s)$ into discrete-time transfer function $H_d(z)$

and then convert into state-space

Sensor bias

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$y_k = Cx_k + v_k$$

So far we assumed w_k and v_k to be zero mean, white

But what if we have a bias in our sensor measurement?

then v_k does not have zero mean.

$$x_{k+1} = Ax_k + Bu_k + w_k \qquad w_k \text{ is zero mean, white with covariance } Q_x.$$

$$y_k = Cx_k + b_k + v_k \rightarrow \text{split into bias + zero mean white noise with Covariance } R$$

↳ bias

Estimate bias by modelling it as a random walk.

$$b_{k+1} = b_k + \xi_k \quad \text{where } \xi_k \text{ is a zero mean white noise with covariance } Q_b$$

$$\begin{bmatrix} x_{k+1} \\ b_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ b_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} w_k \\ \xi_k \end{bmatrix}$$

$$y_k = \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} x_k \\ b_k \end{bmatrix} + v_k$$

$[w_k, \xi_k]^T$ and v_k are zero mean white noises

Apply Kalman filter in the augmented system to estimate states x_k and bias b_k.

→ Covariance of the augmented process noise Q

$$Q = \mathbb{E}\left[\begin{bmatrix} w_k \\ \xi_k \end{bmatrix} \begin{bmatrix} w_k & \xi_k \end{bmatrix} \right] = \mathbb{E}\begin{bmatrix} w_k w_k^T & w_k \xi_k^T \\ \xi_k w_k^T & \xi_k \xi_k^k \end{bmatrix}$$

ξ_k and w_k are uncorrelated

$$\Rightarrow \quad Q = \begin{bmatrix} Q_x & 0 \\ 0 & Q_b \end{bmatrix}$$

Extended Kalman filter

What if the plant model is nonlinear?

$$\dot{x} = f(x,u) + w(t)$$

$$y = h(x) + v(t)$$

process and measurement noise are Gaussian white noise with covariance $\mathbb{E}[w w^T] = Q$, $\mathbb{E}[v v^T] = R$

Discretize the model using Euler discretization

$$x_{k+1} = x_k + \Delta t \, (f(x,u) + w_k$$

Kalman filter requires prediction and correction

→ we can predict

$$\hat{x}_{k|k-1} = \hat{x}_{k-1|k-1} + \Delta t \cdot f(\hat{x}_{k-1|k-1}, u_{k-1}) \,,$$

but how to compute $P_{k|k-1}$? Need A matrix

$$A_c = \left. \frac{\partial f}{\partial x} \right|_{x = \hat{x}_{k-1|k-1}}$$

$$A_k = \mathbb{I} + \Delta t \cdot A_c$$

$$P_{k|k-1} = A_k P_{k-1|k-1} A_k^T + Q$$

→ Update / correction step requires C matrix

$$C_k = \left. \frac{\partial h}{\partial x} \right|_{x = \hat{x}_{k|k-1}}$$

$$L_k = P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R)^{-1}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - h(\hat{x}_{k|k-1}))$$

$$P_{k|k} = (\mathbb{I} - L_k C_k) P_{k|k-1} (\mathbb{I} - L_k C_k)^T + L_k R L_k^T$$

Combined state · parameter estimation

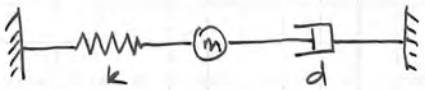
$$\dot{x}_{k+1} = A(\theta) x_k + B u_k + w_k$$

$$y_k = C x_k + v_k$$

$$\theta_{k+1} = \theta_k + \xi_k$$

Augmented system is nonlinear, and requires EKF

Example: Mass - spring - damper.



↳ damping co-efficient unknown.

$$\dot{x}_1 = x_2 + w_1$$

$$\dot{x}_2 = -\frac{k}{m} x_1 - \frac{d}{m} x_2 + \frac{1}{m} u + w_2$$

$$\dot{d} = \xi_1$$

augmented state $z = [x_1, x_2, d]^T$

$$\dot{z} = f(z, u) = \begin{bmatrix} x_2 \\ -\frac{k}{m} x_1 - \frac{d}{m} x_2 + \frac{1}{m} u \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \xi_1 \end{bmatrix}$$

↳ zero mean white noise with covariance $Q \in \mathbb{R}^{3\times 3}$

$$y = x_1$$

Predict

$$\hat{z}_{k|k-1} = \hat{z}_{k-1|k-1} + \Delta t \cdot f(z_{k-1|k-1}, u_{k-1})$$

$$A_c = \left.\frac{\partial f}{\partial x}\right|_{x = \hat{x}_{k-1|k-1}} = \begin{bmatrix} 0 & 1 & 0 \\ -k/m & -d/m & -\frac{x_2}{m} \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = I + \Delta t \cdot A_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \Delta t \cdot \begin{bmatrix} 0 & 1 & 0 \\ -k/m & -d/m & -\frac{x_2}{m} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & \Delta t & 0 \\ -\Delta t \frac{k}{m} & 1 - \Delta t \frac{d}{m} & -\Delta t \frac{x_2}{m} \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Deterministic system

$$x_{k+1} = A x_k + B u_k$$

$$y_k = C x_k$$

We can design the optimal state-feedback control that optimizes

$$\min \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k$$

where $Q = Q^T \succ 0$, $R = R^T \succ 0$ are suitable weighting matrices.

What happens if x and y are random, due to the presence of process and measurement noise?

$\Rightarrow$ DT stochastic system

$$x_{k+1} = A x_k + B u_k + w_k$$

$$y_k = C x_k + v_k$$

with $w_k \sim \mathcal{N}(0, Q_w)$, $v_k \sim \mathcal{N}(0, R_v)$ are zero mean Gaussian white noise

Performance function to minimize:

$$\min \mathbb{E}\left[\sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \right]$$

The solution to this is given by LQG control

Linear Quadratic Gaussian

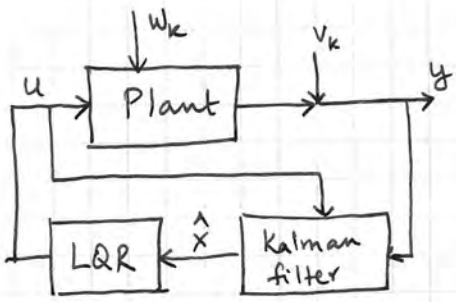
Solution to the LQG problem is surprisingly simple & elegant due to the separation principle

→ design controller gain K for a state-feedback using deterministic LQR, by solving the DARE

$$u_k = -K x_k$$

→ design a Kalman filter to find the optimal estimate $\hat{x}_k$ to minimize the error covariance

and set $u_k = -K \hat{x}_k$ ← solution to the LQG controller.



The closed-loop dynamics for the plant + LQG controller

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$y_k = Cx_k + v_k$$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k)$$

$$u_k = -K\hat{x}_k$$

$$x_{k+1} = Ax_k - BK\hat{x}_k + w_k$$

$$x_{k+1} = Ax_k - BKx_k + BKx_k - BK\hat{x}_k + w_k$$

$$= (A - BK)x_k + BK(x_k - \hat{x}_k) + w_k$$

$$\Rightarrow x_{k+1} = (A - BK)x_k + BKe_k + w_k$$

$$\hat{x}_{k+1} = A\hat{x}_k - BK\hat{x}_k + L(Cx_k + v_k - C\hat{x}_k)$$

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = Ax_k - BK\hat{x}_k + w_k - A\hat{x}_k + BK\hat{x}_k - LCx_k + LC\hat{x}_k - Lv_k$$

$$\Rightarrow e_{k+1} = A(x_k - \hat{x}_k) - LC(x_k - \hat{x}_k) + w_k - Lv_k$$

$$\Rightarrow e_{k+1} = Ae_k - LCe_k + w_k - Lv_k$$

$$= (A - LC)e_k + w_k - Lv_k$$

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{F} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \underbrace{\begin{bmatrix} I & 0 \\ I & -L \end{bmatrix}}_{G} \begin{bmatrix} w_k \\ v_k \end{bmatrix}$$

eigenvalues of the closed loop system are described by the eigenvalues of $(A-BK)$ and $(A-LC)$

↳ consequence of the separation principle

For a deterministic system

$$x_{k+1} = A x_k$$

asymptotic stability implies $x_k = 0$ as $k \to \infty$

we can't say the same for a stochastic system

$$x_{k+1} = A x_k + G w_k \quad , \quad w_k \sim \mathcal{N}(0, Q)$$

instead, we ask for mean-square stability

$$\mathbb{E}[x_k] \to 0 \quad \text{as} \quad k \to \infty$$

$$\mathbb{E}[\|x_k\|^2] < \infty \quad \text{remains bounded.}$$

since $\|x_k\|^2 = \text{trace}(x_k x_k^T)$, this is really about whether the covariance converges to a finite value.

Mean model:

$$\mathbb{E}[x_{k+1}] = A\,\mathbb{E}[x_k] + G\,\mathbb{E}[w_k]$$

$$\mu_{x_{k+1}} = A \mu_{x_k}$$

if all $\text{eig}(A) < 1 \Rightarrow \mu_k \to 0$

Covariance model

$$\mathbb{E}[x_{k+1} x_{k+1}^T] = \mathbb{E}[(Ax_k + Gw_k)(Ax_k + Gw_k)^T]$$

$$= \mathbb{E}[A x_k x_k^T A^T + G w_k w_k^T G^T + A x_k w_k^T G + G w_k x_k^T A]$$

$$= A\,\mathbb{E}[x_k x_k^T] A^T + G\,\mathbb{E}[w_k w_k^T] G^T + 0 + 0$$

$$P_{k+1} = A P_k A^T + G Q G^T$$

if all $\text{eig}(A) < 1$, then there exists a unique positive semi-definite matrix P_∞ that solves the discrete-time Lyapunov equation

$$P_\infty = A P_\infty A^T + \underbrace{G Q G^T}_{:= Q_G}$$

$$\Rightarrow P_\infty = A P_\infty A^T + Q_G$$

Therefore $\mathbb{E}[\|x_k\|^2]$ remains bounded

Apply to the closed-loop LQG system

$$\underbrace{\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix}}_{:= z_{k+1}} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{:= F} \underbrace{\begin{bmatrix} x_k \\ e_k \end{bmatrix}}_{:= z_k} + \underbrace{\begin{bmatrix} I & 0 \\ I & -L \end{bmatrix}}_{:= G} \underbrace{\begin{bmatrix} w_k \\ v_k \end{bmatrix}}_{:= \xi_k}$$

$$\Rightarrow \quad z_{k+1} = F z_k + G \xi_k$$

where $\mathbb{E}[\xi_k] = \mathbb{E}\begin{bmatrix} w_k \\ v_k \end{bmatrix} = 0$

$$\mathbb{E}[\xi_k \xi_k^T] = \mathbb{E}\left[\begin{bmatrix} w_k \\ v_k \end{bmatrix}\begin{bmatrix} w_k^T & v_k^T \end{bmatrix}\right] = \mathbb{E}\begin{bmatrix} w_k w_k^T & w_k v_k^T \\ v_k w_k^T & v_k v_k^T \end{bmatrix}$$

$$\Rightarrow \mathbb{E}[\xi_k \xi_k^T] = \begin{bmatrix} Q_w & 0 \\ 0 & R_v \end{bmatrix}$$

if $\text{eig}(A-BK) < 1$ and $\text{eig}(A-LC) < 1$, then F is Schur which implies that the closed-loop LQG system is mean-square stable.

Random signals

↳ describe by its statistical properties, i.e., mean & covariance.

Random state-space model

$$\dot{x} = Ax + Gw \qquad w \sim \mathcal{N}(0, Q)$$

$$y = Cx + v \qquad v \sim \mathcal{N}(0, R)$$

↳ mean model

$$\dot{\mu}_x = A\mu_x + G\underbrace{\mu_w}_{=0} \Rightarrow \dot{\mu}_x = A\mu_x$$

$$\mu_y = C\mu_x$$

↳ covariance model

$$\dot{C}_x = AC_x + C_xA^T + GQG^T$$

$$C_y = CC_xC^T + R$$

Observer design (Luenberger)

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

error dynamics:

$$\dot{e} = (A-LC)e + Gw - Lv \qquad \text{(Treat this as a random S.S. model)}$$

↳ mean model

$$\dot{\mu}_e = (A-LC)\mu_e$$

↳ covariance model

$$\dot{P} = (A-LC)P + P(A-LC)^T + GQG^T + LRL^T$$

Find L, such that we minimize $\min_L \operatorname{tr}(\dot{P})$ ⇒ kalman filter.