

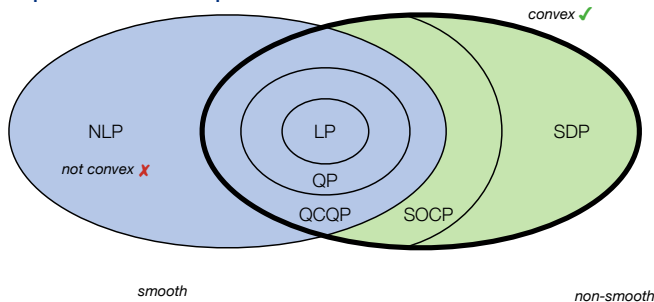
Lecture 2 - Convex Optimization

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Types of optimization problems

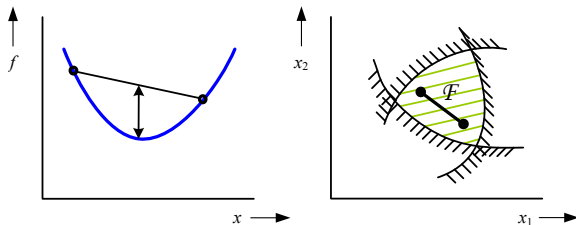


- ▶ If all functions are affine $\Rightarrow$ LP.
- ▶ If quadratic or conic structure appears $\Rightarrow$ QP / QCQP / SOCP / SDP.
- ▶ If **no special structure** (e.g., polynomials, exponentials, trigonometric terms) $\Rightarrow$ **NLP**.

"The great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity"

-R. Tyrell Rockafellar

Convex Optimization Problem



An optimization problem is convex if and only if

1. The feasible domain $\mathcal{F}$ is a convex set
2. The objective function $f(x)$ is a convex function over $\mathcal{F}$

But, what is a convex function? and what is a convex set?

Notation: Line segment

Suppose $x_1 \neq x_2$ are any two points in $\mathbb{R}^n$. Points of the form

$$y = \alpha x_1 + (1 - \alpha)x_2$$

- ▶ with $\alpha \in \mathbb{R}$ form a **line** passing through x_1 and x_2 .
- ▶ with $\alpha \in [0, 1]$ form a (closed) **line segment** between x_1 and x_2 .

Open vs. Closed Sets and Boundaries

Let $S \subset \mathbb{R}^n$ be a set.

- **Interior** : points where some ball stays entirely inside the set

$$\text{int}(S) := \{x \in S \mid \exists r > 0 \text{ such that } B(x, r) \subset S\}$$

where $B(x, r) = \{y \mid \|y - x\| \leq r\}$ is a ball of radius r centered at x in the norm $\|\cdot\|$.

- **Boundary**: A point $x \in \mathbb{R}^n$ is a boundary of the set if every open ball around x contains at least one point in S and at least one point not in S

$$\partial S := \{x \in \mathbb{R}^n \mid \forall r > 0, B(x, r) \cap S \neq \emptyset, \text{ and } B(x, r) \setminus S \neq \emptyset\}$$

i.e., every neighborhood around a boundary point touches both the set and the outside.

- **Closure (includes boundary)**:

$$\text{cl}(S) = \text{int}(S) \cup \partial S$$

Example: Open vs. Closed Sets and Boundaries

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$.

- **Closed set (includes boundary):**

$$\text{cl}(S) = S_{\leq} = \{x \mid g(x) \leq 0\}$$

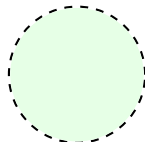
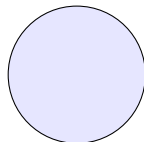
- **Open set (excludes boundary):**

$$\text{int}(S) = S_{<} = \{x \mid g(x) < 0\}$$

- **Boundary:**

$$\partial S = \text{cl}(S) \setminus \text{int}(S) = \{x \mid g(x) = 0\}$$

Optimization convention: Constraints use $g(x) \leq 0$, so boundary points are feasible.



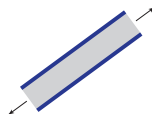
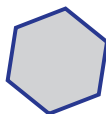
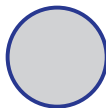
Closed ball ($\|x\| \leq 1$) Open ball ($\|x\| < 1$)

Convex Sets

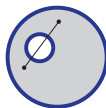
A set $\mathcal{X}$ is convex if and only if for any pair of points x and y in $\mathcal{X}$, any convex combination of x and y lies in $\mathcal{X}$:

$$\mathcal{X} \text{ is convex} \iff \alpha x + (1 - \alpha)y \in \mathcal{X}, \forall \alpha \in [0, 1], \forall x, y \in \mathcal{X}$$

Convex:



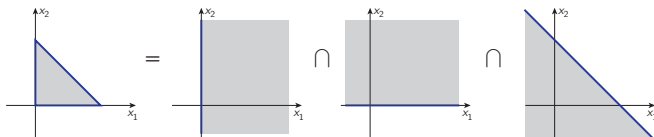
Non-convex:



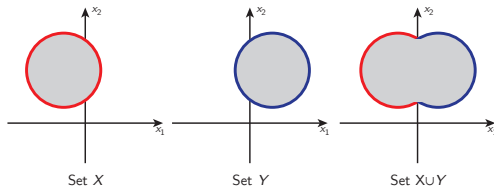
Operations that preserve convexity: Intersection

Theorem

The set resulting from the intersection of two or more convex sets is convex.



This is **not** true for the **union** of convex sets!



Operations that preserve convexity: Affine mappings

- Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine, i.e., $f(x) = Ax + b$

Theorem

The image of a convex set under f is convex

$$\mathcal{X} \subseteq \mathbb{R}^n \text{ is convex} \Rightarrow f(\mathcal{X}) = \{f(x) | x \in \mathcal{X}\} \text{ is convex.}$$

Theorem

The inverse image $f^{-1}(\mathcal{C})$ of a convex set under f is convex

$$\mathcal{C} \subseteq \mathbb{R}^m \text{ is convex} \Rightarrow f^{-1}(\mathcal{C}) = \{x \subseteq \mathbb{R}^n | f(x) \in \mathcal{C}\} \text{ is convex.}$$

Convex combination and conic combination

Convex combination

Convex combination of points $x_1, \dots, x_n$ is any point x of the form

$$x = \alpha_1 x_1 + \dots + \alpha_n x_n, \quad \sum_{i=1}^n \alpha_i = 1, \quad \alpha_i \geq 0$$

A set is convex, if and only if contains every convex combination of its points.

Conic combination

Conic combination of points $x_1, \dots, x_n$ is any point x of the form

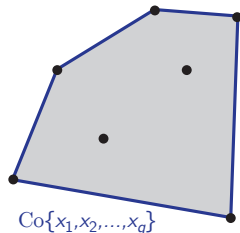
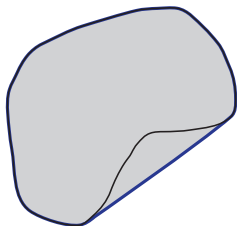
$$x = \alpha_1 x_1 + \dots + \alpha_n x_n, \quad \alpha_i \geq 0$$

A set is a convex cone if and only if it contains all conic combinations of its elements

Convex Hull

The convex hull of a set $\mathcal{X}$ is the set of all convex combinations of points in $\mathcal{X}$:

$$\text{Conv}(\mathcal{X}) := \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid x_i \in \mathcal{X}, \alpha_i \geq 0, i = 1, \dots, n, \sum \alpha_i = 1\}$$

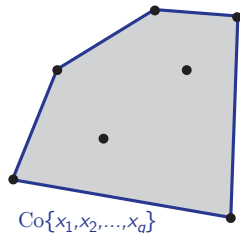
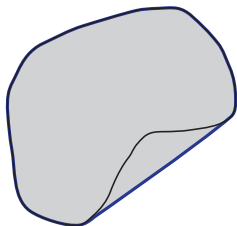


What is the convex hull of a convex set?

Convex Hull

The convex hull of a set $\mathcal{X}$ is the set of all convex combinations of points in $\mathcal{X}$:

$$\text{Conv}(\mathcal{X}) := \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid x_i \in \mathcal{X}, \alpha_i \geq 0, i = 1, \dots, n, \sum \alpha_i = 1\}$$



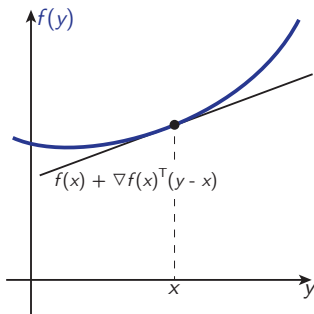
What is the convex hull of a convex set?
The set itself.

First-order Condition for Convexity

A differentiable function $f : \text{dom}(f) \rightarrow \mathbb{R}$ with a convex domain is convex if and only if “any tangent is below”:

$$f(x) + \nabla f(x)^\top (y - x) \leq f(y) \quad \forall x \neq y \in \text{dom}(f).$$

The function f is **strictly** convex if this inequality is strict.

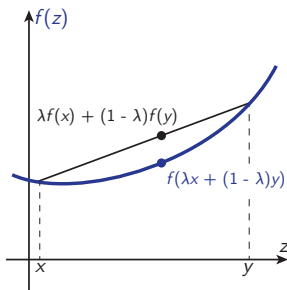


Convex Functions

A function $f : \text{dom}(f) \rightarrow \mathbb{R}$ is convex if and only if its domain $\text{dom}(f)$ is convex and “any secant is above”:

$$\alpha f(x) + (1 - \alpha)f(y) \geq f(\alpha x + (1 - \alpha)y) \quad \forall \alpha \in (0, 1) \quad \forall x \neq y \in \text{dom}(f)$$

The function f is **strictly** convex if this inequality is strict.



Second-order Condition for Convexity

A twice-differentiable function $f : \text{dom}(f) \rightarrow \mathbb{R}$ with a convex domain is convex if and only if the Hessian is positive semi-definite:

$$\nabla^2 f(x) \succeq 0,$$

where the Hessian is defined as

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} = H \triangleq \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

If $\nabla^2 f(x) \succ 0$, then the function is **strictly** convex.

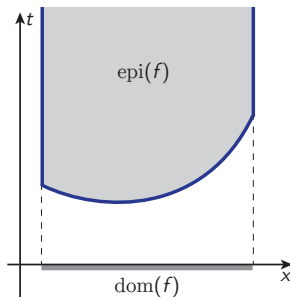
The Epigraph

Definition

The epigraph of a function $f : \text{dom}(f) \rightarrow \mathbb{R}$ is the set

$$\text{epi}(f) = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \middle| x \in \text{dom}(f), f(x) \leq t \right\} \subseteq \text{dom}(f) \times \mathbb{R}$$

A function is convex if and only if its epigraph is a convex set.



Convex Functions

Recap: For the twice-differentiable function $f : \text{dom}(f) \rightarrow \mathbb{R}$ with a convex domain, the following statements are equivalent:

1. The function f is (**strictly**) convex
2. *Secant above*: $\alpha f(x) + (1 - \alpha)f(y) \geq (>) f(\alpha x + (1 - \alpha)y)$
 $\forall \alpha \in (0, 1) \quad \forall x \neq y \in \text{dom}(f)$
3. *Tangent below*: $f(x) + \nabla f(x)^\top (y - x) \leq (<) f(y)$
 $\forall x \neq y \in \text{dom}(f)$
4. *PSD (PD) Hessian*: $\nabla^2 f(x) \succeq (>) 0$
5. Its epigraph is a convex set (no condition for strict convexity!)

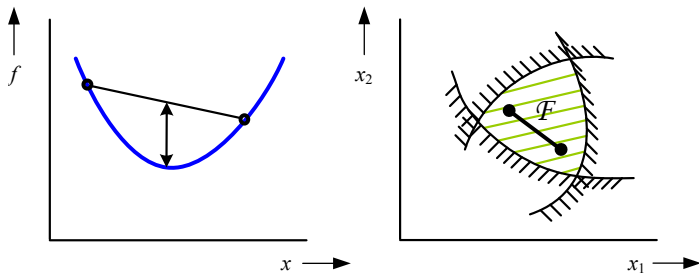
Convexity preserving operations

- ▶ **Non-negative weighted sum:** $\sum_i \theta_i f_i(x)$ is convex if f_i are convex and $\theta_i \geq 0$.
- ▶ **Composition with affine function:** $f(x) = g(Ax + b)$ is convex if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex.
- ▶ **Pointwise maximum:** $f(x) = \max\{f_1(x), \dots, f_m(x)\}$ is convex if all f_i are convex.
- ▶ **Composition:** $h(g(x))$ is convex if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and $h : \mathbb{R} \rightarrow \mathbb{R}$ is convex and non-decreasing
- ▶ **Minimization:** if $g(x, u)$ is jointly convex in (x, u) then

$$f(x) = \inf_u g(x, u)$$

is convex

Convex Optimization Problem



An optimization problem is convex if and only if

1. The feasible domain is a convex set
2. The objective function is a convex function over $\mathcal{F}$

then

1. All KKT points of the problem (*next week*) are global minima
2. and can be computed in polynomial time (i.e., upper bound on time to solve.)

Convex Optimization Problem Standard Form

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad i = 1, \dots, m \\ & a_j^\top x = b_j \quad j = 1, \dots, p. \end{aligned}$$

This problem is convex if

- ▶ The domain $\mathcal{X}$ is a convex set;
- ▶ The objective function f is a convex function;
- ▶ The inequality constraint functions g_i are all convex;
- ▶ The equality constraint functions $h_j(x) = a_j^\top x - b_j$ are affine.

The last two conditions are sufficient, but not necessary.

Convex problem: Any solution is global

Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and let $\mathcal{F} \subseteq \mathbb{R}^n$ be a convex feasible set. Consider the convex optimization problem

$$\min_{x \in \mathcal{F}} f(x).$$

If $x^ \in \mathcal{F}$ is a local minimizer, then x^* is also a global minimizer.*

This can be proved by contradiction!

Proof.

Assume x^* is a local minimizer but not a global minimizer. Then there exists $\bar{x} \in \mathcal{F}$ such that $f(\bar{x}) < f(x^*)$.

Because $\mathcal{F}$ is convex, for any $\alpha \in (0, 1)$ the point $x_\alpha := \alpha\bar{x} + (1 - \alpha)x^*$ also lies in $\mathcal{F}$. By convexity of f ,

$$f(x_\alpha) \leq \alpha f(\bar{x}) + (1 - \alpha)f(x^*) < \alpha f(x^*) + (1 - \alpha)f(x^*) = f(x^*),$$

where the strict inequality uses $f(\bar{x}) < f(x^*)$ and $\alpha > 0$.

Now note that

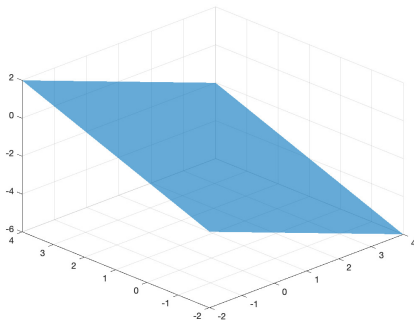
$$\|x_\alpha - x^*\| = \|\alpha(\bar{x} - x^*)\| = \alpha\|\bar{x} - x^*\| \xrightarrow{\alpha \rightarrow 0} 0.$$

Hence, for sufficiently small $\alpha > 0$, x_α lies in any neighborhood of x^* . But we have shown $f(x_\alpha) < f(x^*)$ for all such α , contradicting the assumption that x^* is a local minimizer.

Therefore, the assumption was false, and x^* must be a global minimizer. □

Hyperplanes and halfspaces

- **Hyperplane:** A set of the form $\{x | a^T x = b\}$ ($a \neq 0$)
- **Halfspace:** A set of the form $\{x | a^T x \leq b\}$ ($a \neq 0$)



Hyperplanes and halfspaces are convex!

Convex Feasible Sets

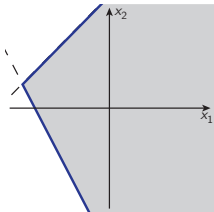
Definition (Polyhedra and Polytopes)

A polyhedron is the intersection of a finite number of closed halfspaces:

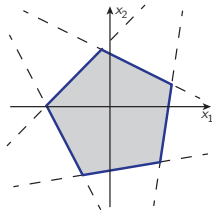
$$\mathcal{X} = \{x | a_1^\top x \leq b_1, a_2^\top x \leq b_2, \dots, a_m^\top x \leq b_m\} = \{x | Ax \leq b\},$$

where $A := [a_1^\top; a_2^\top; \dots; a_m^\top]$, $a_j \in \mathbb{R}^n$, and $b := [b_1; b_2; \dots; b_m]$.

A polytope is a bounded polyhedron.



An (unbounded) polyhedron



A polytope

Polyhedra and polytopes are always convex.

Feasible set = Polyhedron

LP

$$\begin{aligned} \min_x f(x) &:= c^\top x \\ \text{s.t. } h(x) &:= A_{eq}^\top x - b_{eq} = 0, \\ g(x) &:= A^\top x - b \leq 0 \end{aligned}$$

Objective function is **convex**.

QP

$$\begin{aligned} \min_x f(x) &:= \frac{1}{2} x^\top Q x + c^\top x \\ \text{s.t. } h(x) &:= A_{eq}^\top x - b_{eq} = 0, \\ g(x) &:= A^\top x - b \leq 0 \end{aligned}$$

Objective function is **convex** if $Q \succeq 0$

Convex Feasible Sets

Theorem

A c -sublevel set of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, defined as

$$\mathcal{S} = \{x \in \text{dom}(f) \mid f(x) \leq c\}$$

is a convex set.

- Norm balls

$$\mathcal{B}(x_c, r) = \{x \mid \|x - x_c\| \leq r\}$$

- Ellipsoids

$$\mathcal{E} = \{x \mid (x - x_c)^\top P^{-1} (x - x_c) \leq 1\} = \{x_c + Au \mid \|u\|_2 \leq 1\},$$

for any positive definite shape matrix $P \succ 0$.

- Norm cones

$$\mathcal{C} = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$$

Implication: Feasible set of QCQPs and SOCPs are convex.

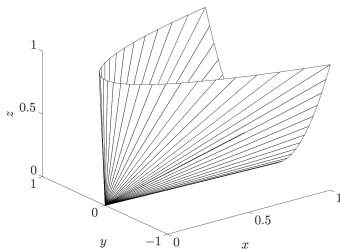
Convex Feasible Sets

$\mathbb{S}^n = \{X \in \mathbb{R}^{n \times n} | X = X^\top\}$ is the set of symmetric $n \times n$ matrices,
 $\mathbb{S}_+^n = \{X \in \mathbb{S}^n | X \succeq 0\}$ is the set of symmetric positive semi-definite matrices,
 $\mathbb{S}_{++}^n = \{X \in \mathbb{S}^n | X \succ 0\}$ is the set of symmetric positive definite matrices.

It holds that $\mathbb{S}_+^n$ is a convex cone.

Example:

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}_+^2 \Rightarrow x \geq 0, z \geq 0, xy \geq y^2$$



Implication: Feasible set of an SDP is convex.

Equivalent Optimization Problems

Two problems are (informally) called **equivalent** if the solution to the one can be (easily) inferred from the solution to the other, and vice versa.

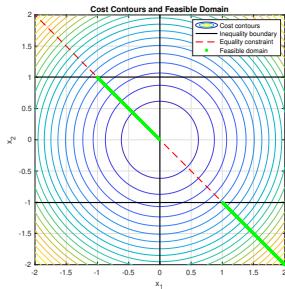
Introduce equality constraints:

$$\begin{array}{ll} \min_x f(A_0x + b_0) & \min_x f(y_0) \\ \text{s.t. } g_i(A_ix + b_i) \leq 0 & \text{s.t. } g_i(y_i) \leq 0 \\ & A_ix + b_i = y_i \end{array}$$

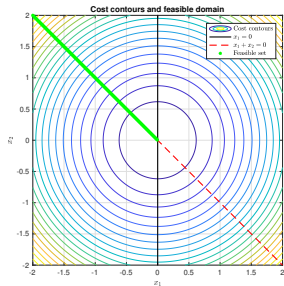
Introducing slack variables for linear inequalities:

$$\begin{array}{ll} \min_x f_0(x) & \min_{x,s} f_0(x) \\ \text{s.t. } A_ix \leq b_i & \text{s.t. } A_ix + s_i = b_i \\ & s_i \geq 0 \end{array}$$

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & \frac{x_1}{1+x_2^2} \leq 0 \\ & (x_1+x_2)^2 = 0 \end{aligned}$$



$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{aligned}$$



Convex relaxation

Convex relaxation works by removing the non-convex part. Given the original problem

$$\min_{x \in \mathcal{X}} f(x)$$

- find a convex function $\hat{f}(x)$ with $\hat{f}(x) \leq f(x)$, $\forall x \in \text{dom}(f)$
- find $\hat{\mathcal{X}} \supseteq \mathcal{X}$ described by linear equalities and convex inequalities

$$\hat{\mathcal{X}} = \{x | g_i(x) \leq 0, i = 1, \dots, m, A^\top x = b\}$$

Then the convex optimization problem

$$\begin{aligned} \min_x \quad & \hat{f}(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & A^\top x - b = 0 \end{aligned}$$

is a convex relaxation of the original problem

- Optimal value of the relaxation is a lower bound on the optimal value of the original problem

Convex relaxation

Convex relaxation $\neq$ Original problem

A convex relaxation in general

- ▶ has a larger feasible set

$$\mathcal{F}_{\text{original}} \subseteq \mathcal{F}_{\text{relax}}$$

- ▶ and creates a lower bound for the optimal value

$$f_{\text{relax}}^* \leq f_{\text{original}}^*$$

- ▶ These are satisfied with equality only when the relaxation is tight (i.e., equivalent)
- ▶ Even when not tight, convex relaxations can give us useful approximations.

Lecture Summary: Convexity in Optimization

- ▶ Optimization problems are classified by the *structure* of the objective and constraints (LP, QP, QCQP, SOCP, SDP, NLP).
- ▶ Convexity is the key property that distinguishes tractable problems from hard ones.
- ▶ Convex sets are preserved under intersection and affine mappings.
- ▶ Convex functions admit multiple equivalent characterizations:
 - ▶ secant above,
 - ▶ tangent below,
 - ▶ positive semidefinite Hessian,
 - ▶ convex epigraph.
- ▶ A convex optimization problem has no spurious local minima: **any solution is global**.

Why Convexity Matters

- ▶ Convex optimization problems can be solved reliably and efficiently, often in polynomial time.
- ▶ LPs, QPs, SOCPs, and SDPs all exploit convex structure for numerical robustness.
- ▶ Many nonconvex problems are approached via *convex relaxations*.
- ▶ Even when not equivalent, convex relaxations:
 - ▶ provide performance bounds (useful in global optimization),
 - ▶ enable scalable computation,
 - ▶ yield useful approximate solutions that can be used for initialization.
- ▶ Convexity is therefore central to modern optimization, control, signal processing, and machine learning.
- ▶ Recognizing and formulating a convex problem is more an art!