

Lecture 7 - Model Predictive Control

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LQR - Linear Quadratic regulator

For a discrete time system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

that optimizes the infinite horizon cost

$$V_{LQR} = \min \sum_{k=0}^{\infty} \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k$$

The LQR control law $\mathbf{u} = -K\mathbf{x}$ and the LQR cost $V_{LQR} = \mathbf{x}^T P \mathbf{x}$ is given as the solution to the Discrete Algebraic Riccati Equation (DARE)

$$P = Q + A^T P A - A^T P B H^{-1} B^T P A$$

which gives the LQR gain

$$K = (R + B^T P B)^{-1} B^T P A$$

Constrained Linear Quadratic control

$$\begin{aligned} \min_{\mathbf{w}} \quad & \sum_{k=0}^{\infty} \mathbf{x}_k^{\top} \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^{\top} \mathbf{R} \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A} \mathbf{x}_k + \mathbf{B} \mathbf{u}_k, \quad k = 0, 1, \dots \\ & \mathbf{x}_{lb} \leq \mathbf{x}_k \leq \mathbf{x}_{ub}, \quad k = 0, 1, \dots \\ & \mathbf{u}_{lb} \leq \mathbf{u}_k \leq \mathbf{u}_{ub}, \quad k = 0, 1, \dots \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned}$$

- ▶ No closed-form solution exists!
- ▶ We no longer have a linear control policy $\mathbf{u} = -\mathbf{K}\mathbf{x}$, but a piece-wise affine control policy.
- ▶ Obtained by solving the QP with a finite horizon
- ▶ Problem is strictly convex if $\mathbf{Q} \succeq 0$ and $\mathbf{R} \succ 0$

Constrained Linear Quadratic control

Quadratic cost

$$\sum_{k=0}^{N-1} \left(\frac{1}{2} \mathbf{x}_k^\top Q \mathbf{x}_k + \frac{1}{2} \mathbf{u}_k^\top R \mathbf{u}_k \right) + \frac{1}{2} \mathbf{x}_N^\top P \mathbf{x}_N$$

Linear system dynamics

$$\mathbf{x}_{k+1} = A \mathbf{x}_k + B \mathbf{u}_k$$

Linear constraints

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} x_{ub} \\ -x_{lb} \\ u_{ub} \\ -u_{lb} \end{bmatrix} = D \mathbf{x}_k + E \mathbf{u}_k \leq b$$

Transcription to QP

We need to convert the linear quadratic control problem into a standard QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{x}_N^\top P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = A \mathbf{x}_k + B \mathbf{u}_k, \quad k = 0, 1, \dots \\ & D \mathbf{x}_k + E \mathbf{u}_k \leq b, \quad k = 0, 1, \dots \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top H \mathbf{w} + h^\top \mathbf{w} \\ \text{s.t.} \quad & F^\top \mathbf{w} - f = 0, \\ & G^\top \mathbf{w} - g \leq 0 \end{aligned}$$

Two approaches:

- ▶ Dense condensed (reduced-space)
- ▶ Sparse non-condensed (full-space)

Dense condensed QP

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}}_{:=\mathcal{A}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{N-2}B & \cdots & AB & B & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}}_{:=\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix}$$

Linear system dynamics

$$\bar{x} = \mathcal{A}x(t) + \mathcal{B}w$$

with $\bar{x} := [x_0 \ x_1 \ \cdots \ x_{N-1} \ x_N]^\top$ and $w := [u_0 \ u_1 \ \cdots \ u_{N-1}]^\top$

Dense condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \bar{\mathbf{x}}^\top \bar{\mathbf{Q}} \bar{\mathbf{x}} + \frac{1}{2} \mathbf{w}^\top \bar{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \bar{\mathbf{x}} = \mathcal{A} \mathbf{x}(t) + \mathcal{B} \mathbf{w} \\ & \bar{\mathbf{D}} \bar{\mathbf{x}} + \bar{\mathbf{E}} \mathbf{w} \leq \bar{\mathbf{b}} \end{aligned}$$

with $\bar{\mathbf{Q}} = \begin{bmatrix} I \otimes Q & 0 \\ 0 & P \end{bmatrix}$, $\bar{\mathbf{R}} = I \otimes R$, $\bar{\mathbf{D}} = I \otimes D$, $\bar{\mathbf{E}} = I \otimes E$, and $\bar{\mathbf{b}} = \mathbf{1} \otimes b$.

Substituting $\bar{\mathbf{x}} = \mathcal{A} \mathbf{x}(t) + \mathcal{B} \mathbf{w}$ and rearranging, we get

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w} \\ \text{s.t.} \quad & [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}] \mathbf{w} \leq \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t) \end{aligned}$$

Dense condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w} \\ \text{s.t.} \quad & [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}] \mathbf{w} \leq \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t) \end{aligned}$$

is in the standard QP form

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top \mathbf{H} \mathbf{w} + \mathbf{h}^\top \mathbf{w} \\ \text{s.t.} \quad & \mathbf{G}^\top \mathbf{w} - \mathbf{g} \leq 0 \end{aligned}$$

with

$$\mathbf{H} := [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}]$$

$$\mathbf{h} := \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B}$$

$$\mathbf{G} := [\bar{\mathbf{D}} \mathcal{B} + \bar{\mathbf{E}}]$$

$$\mathbf{g} := \bar{\mathbf{b}} - \bar{\mathbf{D}} \mathcal{A} \mathbf{x}(t)$$

and decision variables $\mathbf{w} := [\mathbf{u}_0 \quad \mathbf{u}_1 \quad \cdots \quad \mathbf{u}_{N-1}]^\top$

Sparse non-condensed QP

Linear system dynamics

$$\begin{aligned}x_0 &= x(t) \\ Ax_0 + Bu_0 - x_1 &= 0 \\ Ax_1 + Bu_1 - x_2 &= 0 \\ &\vdots \\ Ax_{N-1} + Bu_{N-1} - x_N &= 0\end{aligned}$$

Rearranging in matrix form

$$\underbrace{\begin{bmatrix} I & & & & \\ A & B & -I & & \\ & A & B & -I & \\ & & \ddots & \ddots & \ddots \\ & & & A & B & -I \end{bmatrix}}_{:=F} \underbrace{\begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}}_{:=w} = \underbrace{\begin{bmatrix} x(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{:=f}$$

Sparse non-condensed QP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2} \mathbf{w}^\top H \mathbf{w} \\ \text{s.t.} \quad & F^\top \mathbf{w} - \mathbf{f} = 0, \\ & G^\top \mathbf{w} - \mathbf{g} \leq 0 \end{aligned}$$

$$\text{where } H := \begin{bmatrix} I \otimes \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \\ P \end{bmatrix}, \quad G := [I \otimes [D \quad E]], \quad \mathbf{g} := \mathbf{1} \otimes \mathbf{b},$$

$$F := \begin{bmatrix} I & & & & & & \\ A & B & -I & & & & \\ & & A & B & -I & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A & B & -I \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{x}(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{w} := [\mathbf{x}_0 \quad \mathbf{u}_0 \quad \mathbf{x}_1 \quad \cdots \quad \mathbf{x}_N]^\top$$

Special case: unconstrained LQ

Transcribe to QP using dense-condensed formulation:

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^\top [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathbf{x}(t)^\top \mathcal{A}^\top \bar{\mathbf{Q}} \mathcal{B} \mathbf{w}$$

Necessary and sufficient condition for optimality:

$$\begin{aligned} [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}] \mathbf{w} + \mathcal{B}^\top \bar{\mathbf{Q}}^\top \mathcal{A} \mathbf{x}(t) &= 0 \\ \Rightarrow \mathbf{w} &= [\bar{\mathbf{R}} + \mathcal{B}^\top \bar{\mathbf{Q}} \mathcal{B}]^{-1} \mathcal{B}^\top \bar{\mathbf{Q}}^\top \mathcal{A} \mathbf{x}(t) \end{aligned}$$

Compare with LQR policy: $\mathbf{u} = -(\mathbf{R} + \mathbf{B}^\top \mathbf{P} \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{A} \mathbf{x}(t)$.

Remarks

- ▶ LQR (recursive approach), dense condensed QP, sparse non-condensed QP - all give same numerical solution for unconstrained problem
- ▶ The QP approaches give an open-loop solution, but the LQR (recursive approach) give a closed loop (feedback) solution
- ▶ Feedback is incorporated in the QP approaches by repeatedly solving it (MPC)
- ▶ Linear constraints can be easily added to the QP

Standard stage cost

General quadratic stage cost

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \frac{1}{2} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \mathbf{d}_x^\top \mathbf{x}_k + \frac{1}{2} \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k + \mathbf{d}_u^\top \mathbf{u}_k$$

Setpoint tracking

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \frac{1}{2} (\mathbf{x}_k - \mathbf{x}_r)^\top \mathbf{Q} (\mathbf{x}_k - \mathbf{x}_r) + \dots = \frac{1}{2} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \underbrace{\mathbf{x}_r^\top \mathbf{Q}}_{:= \mathbf{d}_x^\top} \mathbf{x}_k + \dots$$

Penalize input usage

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \dots + \frac{1}{2} \Delta \mathbf{u}_k^\top \mathbf{S} \Delta \mathbf{u}_k$$

Other stage costs

- ▶ It is also possible to define stage costs with ℓ_1 -norm, though this is not very common.
- ▶ With an ℓ_1 -norm, we do not have a QP, but this can be reformulated as a LP (using auxiliary variables).

$$\min_x \|x\|_1 \quad \Rightarrow \quad \begin{array}{ll} \min_{t,x} & \mathbf{1}^T t \\ \text{s.t.} & -t \leq x \leq t \end{array}$$

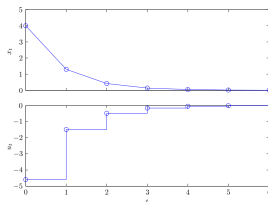
Significance of weights

Weights represent engineering priorities and trade-offs

$$\min \sum_{t=0}^5 q x_{t+1}^2 + r u_t^2$$

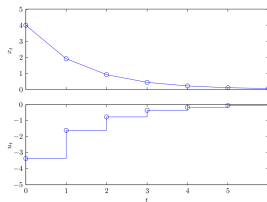
$$\text{s.t.} \quad x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \dots, 5$$

$q = 5, r = 1$



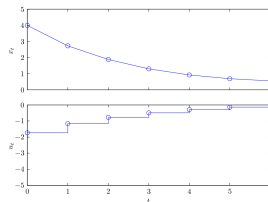
$$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \quad \sum_{t=0}^{N-1} u_t^2 = 23.6$$

$q = 2, r = 1$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \quad \sum_{t=0}^{N-1} u_t^2 = 14.7$$

$q = 1, r = 2$

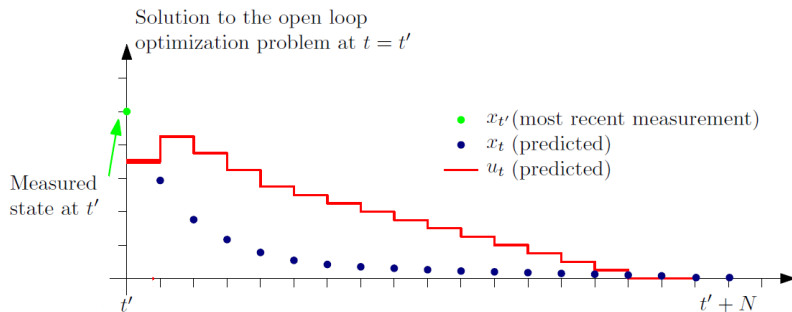


$$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \quad \sum_{t=0}^{N-1} u_t^2 = 5.3$$

Open-loop policy

- Solving QP results in the optimal solution sequence of actions

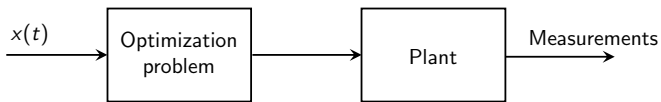
$$[u_0^*, u_1^*, \dots, u_{N-1}^*]^\top$$



- implementing them in open-loop is not robust!

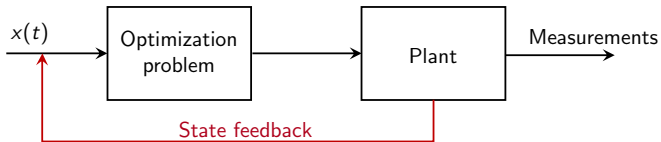
Open-loop policy

$$\pi : \{0, 1, \dots, T - 1\} \rightarrow \mathcal{U}$$



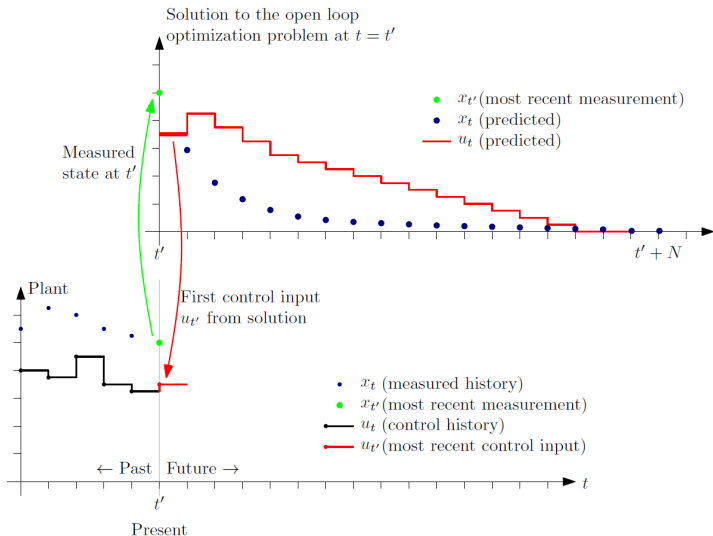
Closed-loop policy

$$\pi : \mathcal{X} \rightarrow \mathcal{U}$$



Model Predictive Control

$$\pi_{\text{mpc}}(x) = u_0^*$$



Model Predictive Control

1. At each time t , solve QP

$$\min_w \mathbf{x}_N^\top P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top Q \mathbf{x}_k + \mathbf{u}_k^\top R \mathbf{u}_k$$

$$\text{s.t. } \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad k = 0, 1, \dots$$

$$D\mathbf{x}_k + E\mathbf{u}_k \leq b, \quad k = 0, 1, \dots$$

$$\mathbf{x}_0 = \mathbf{x}(t)$$

to obtain optimal sequence

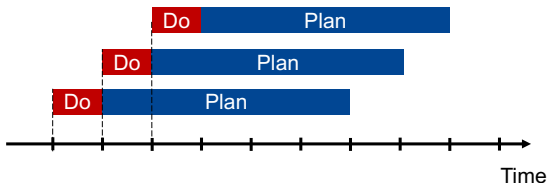
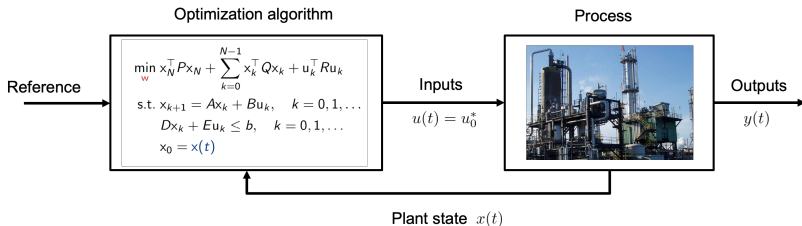
$$[\mathbf{u}_0^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{N-1}^*]^\top$$

2. Implement first action $\mathbf{u}(t) = \pi_{\text{mpc}}(\mathbf{x}(t)) = \mathbf{u}_0^*$
3. Measure next state $\mathbf{x}_{t+1}$, and go to step 1.

This introduces feedback!

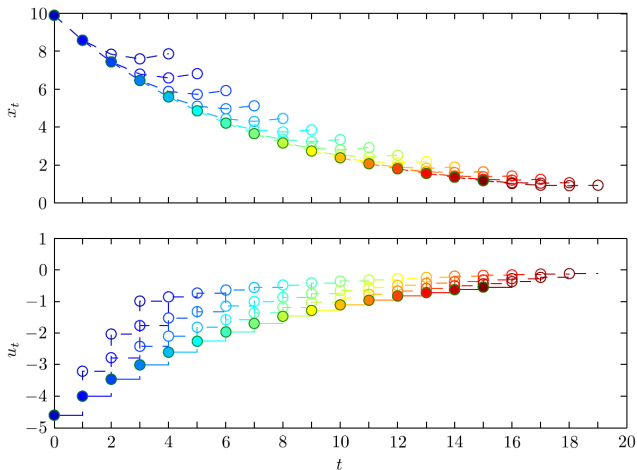


Model Predictive Control

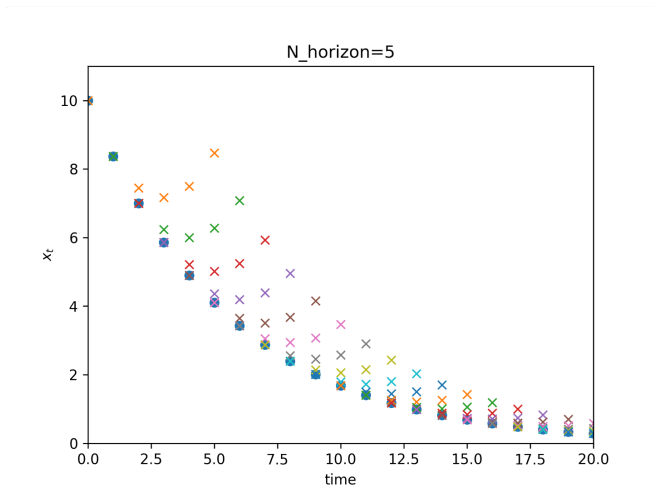


Consequence of finite horizon

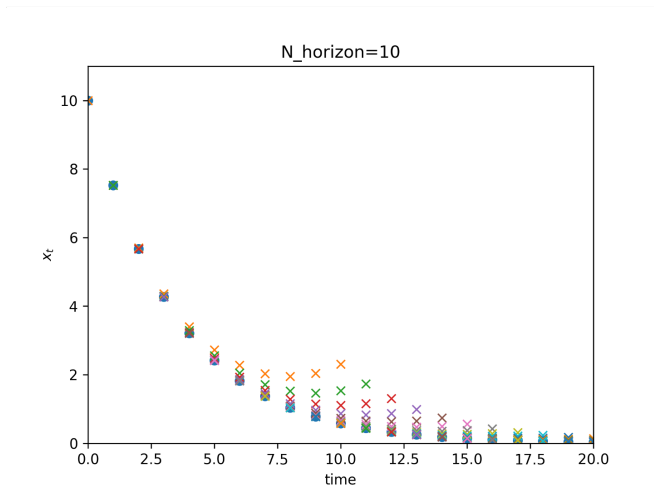
Closed-loop trajectories different from open-loop (optimized) trajectories!



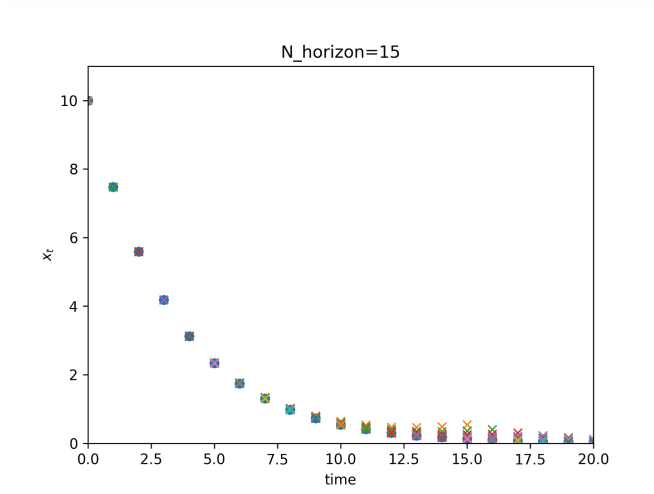
Consequence of finite horizon



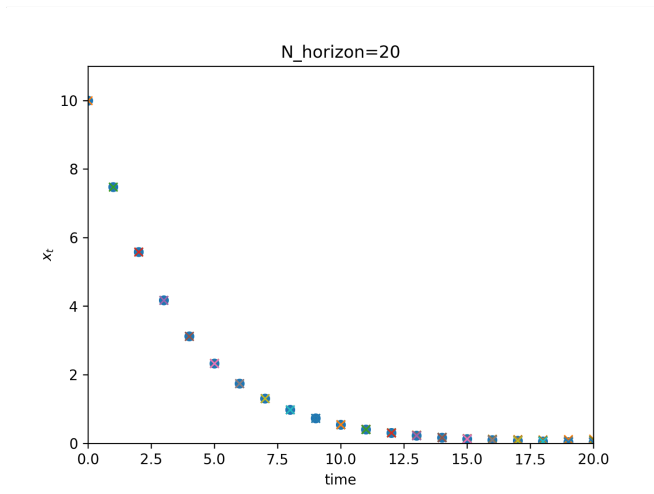
Consequence of finite horizon



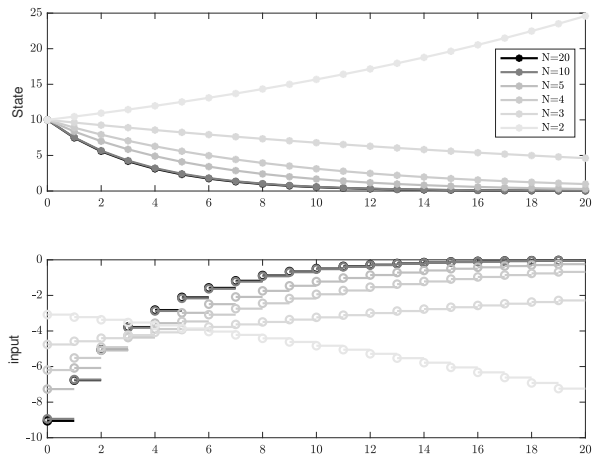
Consequence of finite horizon



Consequence of finite horizon



Closed-loop solution



Small horizon can lead to unstable closed-loop performance!!

Stability of MPC

Infinite horizon: LQR

- ▶ LQR is the solution to the infinite horizon problem

Choose N large enough!

- ▶ Can show theoretically that stability is guaranteed for N large enough, but difficult/conservative to compute this limit
- ▶ Too large N can be computationally expensive (large QP!)

what is *large enough* in practice?

- ▶ Rule of thumb 1 (simple): Choose N longer than dominating dynamics
- ▶ Rule of thumb 2 (better): Choose N large enough such that open-loop predictions resemble closed-loop (test in simulations!)

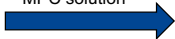
If N is not large, then we need good estimate of the cost-to-go as our terminal cost.

MPC optimality does not imply stability

$$\min \sum_{t=0}^1 \frac{1}{2} x_{t+1}^2 + \frac{r}{2} u_t^2$$

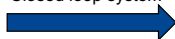
$$\text{s.t.} \quad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1$$

MPC solution

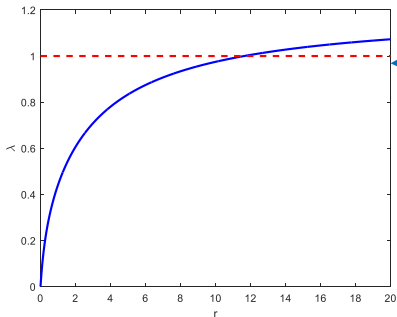


$$u_t = -\frac{1.2 + 2.928r}{1 + 3.44r + r^2} x_t$$

Closed loop system



$$x_{t+1} = \left(1.2 - \frac{1.2 + 2.928r}{1 + 3.44r + r^2} \right) x_t$$



- For given N , design Q and R such that MPC is stable .
- But usually we want to design (tune) Q and R for performance!

Terminal ingredients

Constraints can be active when far from setpoint, but become irrelevant close to setpoint In other words: MPC "reduces" to (same solution as) LQR when close to setpoint

- ▶ Add a terminal cost $x_N^\top P x$, where P is the solution to the DARE.

$$P = Q + A^\top P A - A^\top P B H^{-1} B^\top P A$$

- ▶ Add a terminal constraint $x_N \in \mathcal{X}_f$.

Define a set $\mathcal{X}_f$ such that for all $x \in \mathcal{X}_f$ such that $\mathcal{X}_f$ is control invariant under $u = -Kx$

- ▶ The LQR control law $u = -Kx$ satisfies constraints
- ▶ The next state also remains in $\mathcal{X}_f$

$$\mathcal{X}_f = \{x \in \mathcal{X} \mid -Kx \in \mathcal{U}, (A - BK) \in \mathcal{X}_f\}$$

At the end of the horizon, the system must be in a region where LQR can take over safely.

Feasibility

Is there always a feasible solution to the MPC open-loop optimization problem?

- ▶ Not necessarily - state constraints may become infeasible!
- ▶ Soft constraints by adding slack variables

Offset-free MPC - Adding integral action

- ▶ An unmodeled disturbance (plant-model mismatch) will give offset in the controlled variables
- ▶ But, no Integral action in MPC !

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + B_d d_k \\y_k &= Cx_k + C_d d_k \\d_{k+1} &= d_k\end{aligned}$$

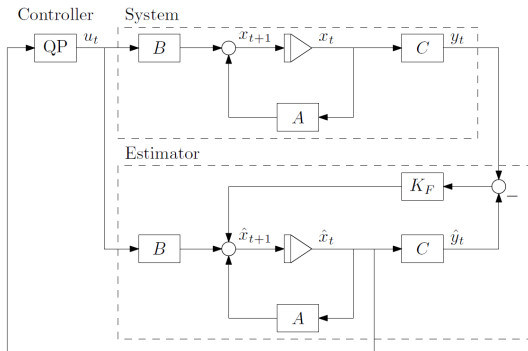
Use the augmented model with a state estimator to estimate $\hat{x}_k$ and $\hat{d}_k$

$$\begin{aligned}\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + w_k \\ y_k &= [C \quad C_d] \begin{bmatrix} x_k \\ d_k \end{bmatrix}\end{aligned}$$

Output feedback MPC

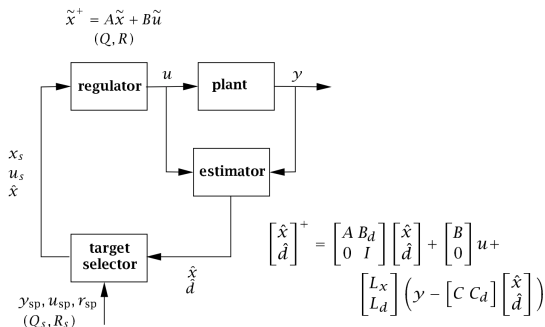
The MPC policy $u(t) = \pi_{\text{mpc}}(x(t))$ is a state-feedback controller!

- Not all states are measured.
- Estimate using Kalman filter / Luenberger observer

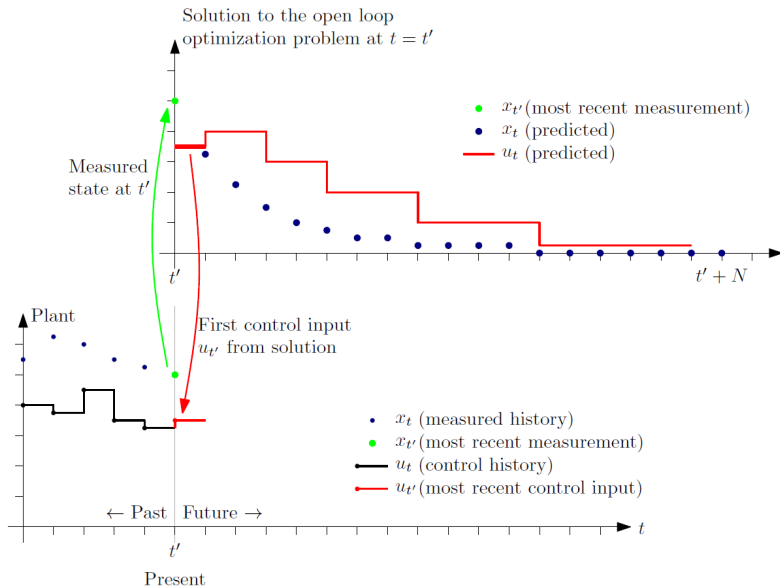


Setpoint tracking

- We want $r_k \rightarrow r_{sp}$, where $r_k = Hy_k$
- NB! controlled variables $r = HCx$ may be different from measured variables $y = Cx$



Complexity reduction: Input blocking



Target Calculator for setpoint tracking

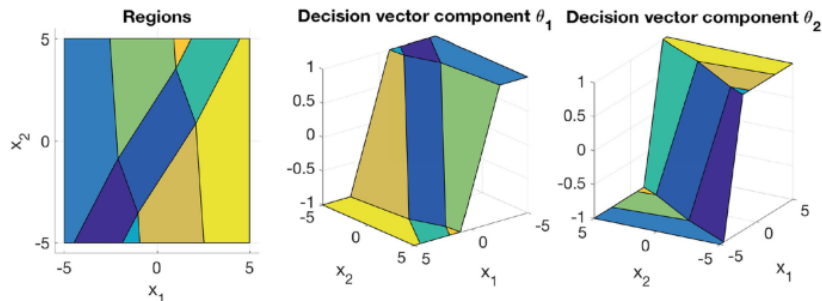
$$\begin{aligned} \min_{\mathbf{x}_s, \mathbf{u}_s} \quad & \frac{1}{2}(\mathbf{C}\mathbf{x}_s - y_{sp})^\top \mathbf{Q}(\mathbf{C}\mathbf{x}_s - y_{sp}) + \frac{1}{2}(\mathbf{u}_s - \mathbf{u}_{iv})^\top \mathbf{R}(\mathbf{u}_s - \mathbf{u}_{iv}) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{HC} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_s \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ r_{sp} \end{bmatrix} \\ & y_{lb} \leq \mathbf{C}\mathbf{x}_s \leq y_{ub} \\ & \mathbf{u}_{lb} \leq \mathbf{u}_s \leq \mathbf{u}_{ub} \end{aligned}$$

Using $(\mathbf{x}_s, \mathbf{u}_s)$, formulate zero-state regulation problem with deviation variables

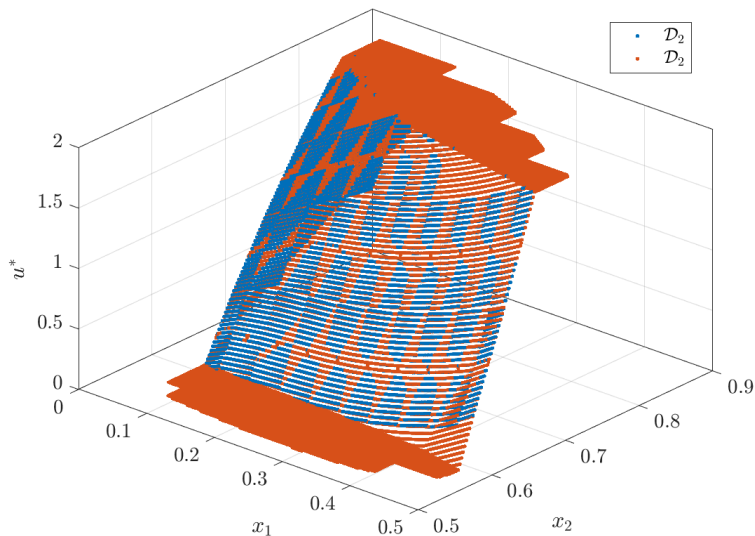
- ▶ $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_s$
- ▶ $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_s$

Complexity reduction: Explicit MPC

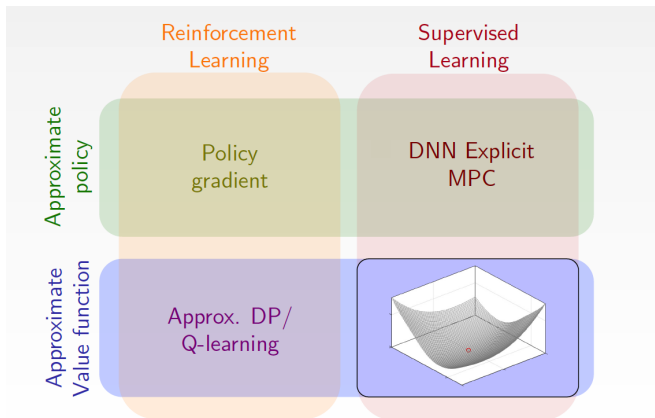
Precompute the piecewise affine mapping using multiparametric programming.



Complexity reduction: Approximate Explicit MPC



Complexity reduction: Horizon compression



Myopic MPC $\left\{ \begin{array}{l} \min_{u \in \mathcal{U}} \ell(x(t), u) + V_f(x^+) \\ \text{s.t. } x^+ = f(x(t), u) \in \mathcal{X} \end{array} \right.$

When to use MPC instead of PID?

PID is usually sufficient when:

- ▶ Single-input single-output (SISO) systems.
- ▶ Minimal interaction with other control loops.
- ▶ Very low implementation and maintenance complexity required.

MPC becomes preferable when:

- ▶ **Strong coupling between variables:** each input affects multiple outputs.
- ▶ **Multiple objectives must be balanced:** it is not clear how to use the available inputs to meet all goals.
- ▶ **Future events matter:** known reference changes or disturbances can be anticipated.

Summary - Mind map

Open loop policy

$$\{0, 1, \dots\} \rightarrow \mathcal{U}$$

$$\begin{aligned} \min_{\mathbf{w}} \quad & \sum_{k=0}^{\infty} \ell(\mathbf{x}_k, \mathbf{u}_k) \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) \\ & \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k) \leq 0 \\ & \mathbf{x}_0 = \mathbf{x}(t) \end{aligned}$$

(no feedback)



Control Policy $\pi(\mathbf{x})$

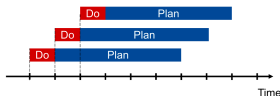
$$\pi : \mathcal{X} \rightarrow \mathcal{U}$$

$$k = N-1, N-2, \dots, 0$$

$$J_k^*(\mathbf{x}_k) = \min_{\mathbf{u}_k \in \mathcal{U}_k(\mathbf{x}_k)} \underbrace{\ell_k(\mathbf{x}_k, \mathbf{u}_k)}_{\text{cost now}} + \underbrace{J_{k+1}^*(f(\mathbf{x}_k, \mathbf{u}_k))}_{\text{cost-to-go}}$$

Compute using
backward DP recursion
(in general intractable, except for LQR)

Model Predictive Control



Summary - Mind map

Choose N to ensure closed-loop system is stable

Choose Q and R to trade-off engineering priorities

$$\min \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + \frac{1}{2} x_N^T P x_N$$

$$\text{s.t. } x_{k+1} = A x_k + B u_k$$

$$x_{lb} \leq x_k \leq x_{ub}$$

$$u_{lb} \leq u_k \leq u_{ub}$$

$$x_0 = x(t)$$

$$x_N \in \mathcal{X}_f$$

LQR cost
from DARE

region where
MPC behaves like LQR

Transcribe to QP

► Dense condensed QP

$$w = [u_0, u_1, \dots, u_{N-1}]^T$$

► Sparse noncondensed QP

$$w = [x_0, u_0, x_1, \dots, u_{N-1}, x_N]^T$$

$$\min_w \frac{1}{2} w^T H w + h^T w$$

$$\text{s.t. } F w = f$$

$$G w \leq g$$

Solve QP
to get u_0^*

• Feasibility

- Soft constraints with slack variables

- $x_{lb} + s \leq x_k \leq x_{ub} + s$ • $s \geq 0$
- Add exact penalty $\rho^T \|s\|_1$

• Setpoint tracking

$$\min_{x_s, u_s} \frac{1}{2} (C x_s - y_{sp})^T Q (C x_s - y_{sp}) + \frac{1}{2} (u_s - u_{iv})^T R (u_s - u_{iv})$$

$$\text{s.t. } \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r_{sp} \end{bmatrix}$$

$$y_{lb} \leq C x_s \leq y_{ub}$$

$$u_{lb} \leq u_s \leq u_{ub}$$

• Disturbance rejection

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix}$$

Formulate zero-state regulation
using deviation variables

State
estimator

measurements
 y

Plant